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# EVALUATION OF THE ENERGY CHARACTERISTICS OF NONSTATIONARY BROADBAND RANDOM PROCESSES ACCORDING TO INDIVIDUAL REALIZATIONS

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## ANNOTATION

Based on the connection between stationary (uniform), in the broad sense, random processes and white noise, the theory of estimation of the intensity of white noise, according to individual realizations, and of their practical application to analysis of digital computer modeled and experimental nonstationary, broad-band random processes, are examined in this work.

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## Introduction /3*

For stationary, in the broad sense, random processes, a detailed theory has been developed, of estimates of the energy (quadratic) characteristics, such as the correlation function and spectral density, according to individual realizations [19, 20, 22]. The results of this theory are extensively applied in solution of practical problems, both with use of special analog instruments and with digital computer application [1, 9, 12, 17].

However, completely stationary processes are not encountered in nature. More than that, in many cases, actual processes differ strongly from this idealized scheme.

Spectral density does not exist in a nonstationary random process. Therefore, for a description of the energy distribution over the spectrum, new characteristics have to be introduced, which depend not only on frequency, but on time. For example, the instantaneous spectrum, defined by the relationship

$$W_x(f,t) = \lim_{T\to\infty} 2 \frac{\partial}{\partial t}\left\{ \mathcal{E}\left[ |S_{xT}(f,t)|^2 \right] \right\}, \tag{B.1}$$

where

$$S_{xT}(f,t) = \int_{-T/2}^{t} x(t)\, e^{-i2\pi f t}\, dt \tag{B.2}$$

is obtained by transformation of realization x(t), can be used as such a characteristic of a nonstationary random process X(t) [23, 24]. The instantaneous spectrum is connected with the correlation function by a certain analog relationship of Wiener-Khintchine:

$$W_x(f,t) = 4\int_0^\infty K_x(t,t-\tau)\cos 2\pi f\tau \, d\tau ,$$
$$K_x(t,t-\tau) = \int_0^\infty W_x(f,t)\cos 2\pi f\tau \, df . \tag{B.3}$$

Systematic investigation of the application of this characteristic, which we conducted, showed that the actual (i.e., with small shifts and dispersion) estimates of the instantaneous

*Numbers in the margin indicate pagination in the foreign text.

spectrum according to individual realizations can be obtained, only when the process does not differ too strongly from stationary. Moreover, in giving a very graphic representation of the frequency distribution of energy with time, the instantaneous spectrum $W_x$ (f, t) is practically unsuitable for determination of the dispersion of the response of a linear system to an essentially nonstationary perturbation. The same remarks can be made with respect to other analogous characteristics [1, 25]. The significant feature here is the fact that all these characteristics depend on two variables f and t.

In connection with this, the thought arises of examination of another idealized limiting case, when $W_x$(f, t) does not depend (or depends slightly) on frequency f. We arrive at this phase, by examining the so-called white noise, i.e., an idealized random process with uncorrelated values, in which /4

$$K_x(t_1, t_2) = b_x(t_1)\,\delta(t_1 - t_2)\,, \tag{B.4}$$

where $\delta(t)$ is the Dirac delta function, and $b_x(t)$ is the white noise intensity.

By substitution of (B.4) in (B.3) and considering that

$$\int_0^\infty \varphi(t)\,\delta(t)\,dt = \frac{1}{2}\varphi(0)\,,$$

we obtain

$$W_x(f, t) = 4\int_0^\infty b_x(t)\,\delta(\tau)\cos 2\pi f\tau\,d\tau = 2\,b_x(t)\,. \tag{B.5}$$

The white noise intensity, for example, permits easy determination of the correlation function of the linear system response. Let the response of system Y(t) to the action of random process X(t) be given by the relationship

$$Y(t) = \int_{-\infty}^{+\infty} h(t, \xi)\,X(\xi)\,d\xi\,. \tag{B.6}$$

Then

$$K_Y(t_1, t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_1, \xi_1)\,h(t_2, \xi_2)\,K_x(\xi_1, \xi_2)\,d\xi_1 d\xi_2 \tag{B.7}$$

and, for white noise, by substitution of (B.4) here, we obtain

$$K_Y(t_1, t_2) = \int_{-\infty}^{+\infty} h(t_1, \xi_1)\,h(t_2, \xi_1)\,b_x(\xi_1)\,d\xi_1\,. \tag{B.8}$$

White noise is the limiting form of actual broadband nonstationary processes, the correlation functions of which are close to (B.4), in the sense that relationship (B.8) is approximately satisfied, with a certain function $b_x(t)$. This function can be approximately interpreted as the effective intensity of a broadband process or, in accordance with (B.5), as half of the instantaneous spectrum.

Certain necessary information is presented in the first chapter of this work, from the theory of generalized functions and the theory of random processes and, based on the connection existing between stationary (uniform), in the broad sense, processes and white noise, the theory of estimates of white noise intensity according to individual realizations is examined.

In the second chapter, questions are studied which are connected with the application of these estimates to analysis of digital computer model and experimental nonstationary, broadband random processes.

## CHAPTER I

## THEORY OF ESTIMATES OF WHITE NOISE INTENSITY

### 1. Some Information from Theory of Generalized Functions

White noise is an idealized random process, and it cannot be /5
realized experimentally, since infinite power is required to maintain it. This fact is expressed mathematically, in that both realization of the process and certain statistical characteristics of it (for example, the correlation function) cannot be described by means of ordinary functions. Therefore, the simplest and most consistent theory of such process can be stated with the use of the apparatus of generalized functions.

For the purpose of this work, what physical meaning should be put into the concept of white noise and what mathematical algorithms permit an actual determination of the quantitative characteristics of the process must be quite clearly represented. For this we use certain simplest results of the theory of generalized functions[2, 4-6]. In order not to overload the work with a large amount of special concepts and terms, we restrict ourselves to only a qualitative description of these results, without resorting to strict and detailed mathematical formulations.

We analyze set $\mathcal{K}_1$ of all functions $\psi(\xi)$, each of which has continuous derivatives of all orders and is finite, i.e., it reverts to zero, outside a certain limited interval (of its own, for each function $\psi(\xi)$). The sequence $\psi_1(\xi)$, $\psi_2(\xi)$, . . . , $\psi_n(\xi)$, . . . is considered to be convergent if all functions of the sequence revert to zero outside the same interval and converge uniformly toward a limiting function, just like their derivatives of any order. With this definition of convergence of a sequence, functions $\psi(\xi)$ are called basic, and the set of them $\mathcal{K}_1$, basic function space. Space $\mathcal{K}_1$ is evidently linear, i.e., together with $\psi_1(\xi)$ and $\psi_2(\xi)$, functions $\alpha\psi_1(\xi) + \beta\psi_2(\xi)$ belong to space $\mathcal{K}_1$, for any constant $\alpha$ and $\beta$. We note that the analytical functions of a real variable are not included in space $\mathcal{K}_1$, since they cannot identically revert to zero in a finite interval. An example of a function, which is infinitely differentiable and which reverts to zero outside the interval $-a < \xi < a$, can be

$$\psi(\xi, a) = \begin{cases} e^{-\frac{a^2}{a^2-\xi^2}} & \text{for } |\xi| < a, \\ 0 & \text{for } |\xi| \geqslant a. \end{cases} \tag{1.1}$$

Now, let a locally integrable $f(\xi)$ be assigned, i.e., the function of the absolute integrand in any finite interval. Then, by means of $f(\xi)$, a linear functional can be assigned

$$\langle f, \psi \rangle = \int_{-\infty}^{+\infty} f(\xi)\psi(\xi)\,d\xi \, , \tag{1.2}$$

which each function $\psi(\xi)$ of space $\mathcal{K}_1$ places in the corresponding number $\langle f, \psi\rangle$. In (1.2), integration actually takes place over a finite interval, in which a given function $\psi(\xi)$ differs from zero. Let the sequence of basic functions $\psi_1(\xi)$, $\psi_2(\xi)$, . . . . , $\psi_n(\xi)$, . . . converge in $\mathcal{K}_1$ towards basic function $f(\xi)$. Since local integrability of function $f(\xi)$ and the nature of convergence in $\mathcal{K}_1$ permit transition to the limit in (1.2), under the integral sign,

$$\begin{aligned} \lim_{n\to\infty} \langle f, \psi_n \rangle &= \lim_{n\to\infty} \int_{-\infty}^{+\infty} f(\xi)\psi_n(\xi)\,d\xi = \\ &= \int_{-\infty}^{+\infty} f(\xi)\psi(\xi)\,d\xi = \langle f, \psi \rangle \, . \end{aligned} \tag{1.3}$$

In this manner, functional (1.2) is not only linear, but continuous, in the basic space. /6

It can be shown that the values of functional (1.2) in space $\mathcal{K}_1$ unambiguously determine function $f(\xi)$, i.e., any locally integrable function can be fixed by its values, for each value of the argument, and it can be fixed, by means of an infinite set of numbers (1.2), corresponding to all possible basic functions. Formula (1.2) gives a very particular form of linear continuous functionals in space $\mathcal{K}_1$, which usually are called regular. By generalization of only what has been said, any linear continuous functional, defined in basic space $\mathcal{K}_1$, is identified with a certain generalized function f, and they are written in the form of the symbolic equality

$$\langle f, \psi \rangle = \int_{-\infty}^{+\infty} f(\xi)\psi(\xi)\,d\xi \, . \tag{1.4}$$

In the general case, a generalized function f cannot be fixed by its values for each value of the argument and, consequently, it is not a function in the ordinary sense. The most well-known example of a generalized function is the Dirac $\delta$ function, which is defined by the relationship

$$\langle \delta, \psi \rangle = \int_{-\infty}^{+\infty} \delta(\xi)\psi(\xi)\,d\xi = \psi(0) \, . \tag{1.5}$$

It should be noted that linear, continuous functionals (and, consequently, generalized functions), defined in $\mathcal{K}_1$, usually can be continued to broader classes (spaces) of functions. Thus, for example, $\delta$ function (1.5) has been defined for any function bounded almost everywhere, fixed by its values at each point.

Summation and multiplication operations by number, for generalized functions, are defined by the relationships:

$$\begin{aligned} &\langle f_1 + f_2, \psi \rangle = \langle f_1, \psi \rangle + \langle f_2, \psi \rangle, \\ &\langle \alpha f, \psi \rangle = \alpha \langle f, \psi \rangle = \langle f, \alpha \psi \rangle. \end{aligned} \tag{1.6}$$

The product of generalized functions is not determined in the general case, but multiplication of generalized functional f by function $a(\xi)$, which is infinitely differentiable, is given by the formula

$$\langle a f, \psi \rangle = \langle f, a \psi \rangle. \tag{1.7}$$

The limiting transition is determined in the following manner. The sequence of generalized functions $f_1$, $f_2$, $f_3$, . . . , $f_n$ . . . , by definition, converges to generalized functional f

$$\lim_{n \to \infty} f_n = f, \tag{1.8a}$$

if, for any basic function

$$\lim_{n \to \infty} \langle f_n, \psi \rangle = \langle f, \psi \rangle. \tag{1.8b}$$

It can be shown that, when a sequence of generalized functions /7 converges, its limit is determined unambiguously, and it always is a generalized function. The operation of the limiting transition is linear, i.e., from the condition

$$\lim_{n \to \infty} f_{1n} = f_1, \quad \lim_{n \to \infty} f_{2n} = f_2$$

it follows that

$$\lim_{n \to \infty} (\alpha f_{1n} + \beta f_{2n}) = \alpha f_1 + \beta f_2, \tag{1.9}$$

where $\alpha$ and $\beta$ are any numbers or infinitely differentiable functions.

If a sequence of locally integrable functions $f_1(\xi)$, $f_2(\xi)$, $f_3(\xi)$, . . . , $f_n(\xi)$, . . . , converging towards locally integrable function $f(\xi)$, such that in (1.?), one can proceed to the limit under the integral sign, for any basic function

$$\lim_{n\to\infty} \langle f_n, \psi \rangle = \langle f, \psi \rangle ,$$

i.e., the sequence converges, in the sense of generalized functions. This takes place, for example, if $f_n(\xi) \to f(\xi)$ almost everywhere, and $|f_n(\xi)|$ is bounded by a fixed constant in each finite interval.

The following proposition is important for practical applications. Any generalized function (functional) can be obtained, as the limit of ordinary functions (regular functionals). As an example, we introduce the so-called delta-form sequence, i.e., sequences of ordinary functions, which are convergent in the meaning of generalized functions to $\delta$-function (1.3). In order for sequence $f_1(\xi)$, $f_2(\xi)$, . . . , $f_n(\xi)$, . . . to be delta-form, the following conditions must be satisfied:

a. for any $U > 0$ and $|a| \le U$, $|b| \le U$, the quantities

$$\left| \int_a^b f_n(\xi)\, d\xi \right| \le C \qquad (1.10)$$

are bounded by constant C, independent of a, b and n;

b. for any fixed a and b, not equal to zero,

$$\lim_{n\to\infty} \int_a^b f_n(\xi)\, d\xi = \begin{cases} 0 & \text{for } a<b<0 \text{ or } 0<a<b, \\ 1 & \text{for } a<0<b. \end{cases} \qquad (1.11)$$

One of the most common ways to construct a delta-form sequence consists of the following. Let $w(\xi)$ be any function, which satisfies the conditions

$$\begin{aligned} &1. \quad w(\xi) = w(-\xi) ; \\ &2. \quad \lim_{A\to\infty} \int_{-A}^{A} w(\xi)\, d\xi = 1 ; \\ &3. \quad \int_{-\infty}^{+\infty} |w(\xi)|^2\, d\xi < \infty . \end{aligned} \qquad (1.12)$$

Let us examine a sequence of functions with the common term

$$w_{L_n}(\xi) = L_n w(L_n \xi), \quad \lim_{n\to\infty} L_n = \infty. \tag{1.13}$$

According to the second condition of (1.12), for $a < 0 < b$, /8

$$\lim_{n\to\infty} \int_a^b w_{L_n}(\xi)\, d\xi = \lim_{n\to\infty} \int_{L_n a}^{L_n b} w(\xi)\, d\xi = 1,$$

and, for $a < b < 0$ or $0 < a < b$,

$$\lim_{n\to\infty} \int_a^b w_{L_n}(\xi)\, d\xi = \lim_{n\to\infty} \int_{L_n a}^{L_n b} w(\xi)\, d\xi = 0.$$

Further, according to the third condition of (1.12), $w(\xi)$ is integrable in each finite interval. We form the nonnegative continuous function

$$F(\eta) = \left| \int_0^\eta w(\xi)\, d\xi \right|.$$

From the first and second conditions of (1.12), it is seen that function $F(\eta)$ is bounded on all axes $-\infty < \eta < +\infty$, by a certain constant B and, consequently,

$$\left| \int_a^b w_{L_n}(\xi)\, d\xi \right| = \left| \int_{L_n a}^{L_n b} w(\xi)\, d\xi \right| \leqslant F(L_n b) + F(L_n a) \leqslant 2B,$$

where B is independent of a, b and n. In this manner, for $w_{L_n}(\xi)$, conditions (1.10) and (1.11) are fulfilled, and the sequence is delta-form. The set of delta-form sequences constructed encompasses everything usually used in practice. From the reasoning presented, it is seen that the third condition of (1.12) can be replaced by the requirement of integribility of the function in any finite interval.

We now examine the determination of more complex operations with generalized functions. All these operations result by distribution to generalized functions of the corresponding operations for common functions, expressed in the language of functionals. Let $f(\xi)$ be a continuous function, having a continuous derivative $f'(\xi)$. Then, by partial integration, considering that each basic function $\psi(\xi)$ reverts identically to zero outside a certain interval, we obtain

$$\langle f', \psi \rangle = \int_{-\infty}^{+\infty} f'(\xi)\psi(\xi)\,d\xi = -\int_{-\infty}^{+\infty} f(\xi)\psi'(\xi)\,d\xi = \langle f, -\psi' \rangle,$$

where $\psi'(\xi)$ also is a basic function. In accordance with this, derivative $f'$ of generalized function $f$ is defined by the relationship

$$\langle f', \psi \rangle = \langle f, -\psi' \rangle \tag{1.14a}$$

or, in symbolic form,

$$\langle f', \psi \rangle = -\int_{-\infty}^{+\infty} f(\xi)\psi'(\xi)\,d\xi . \tag{1.14b}$$

It can be shown that the functional defined by formula (1.14) is linear and continuous. In this manner, in distinction from ordinary functions, all generalized functions have a generalized derivative and, more than that, they are infinitely differentiable.

Further, in distinction from ordinary functions, a converging sequence of generalized functions can always be differentiated termwise. Actually, let sequence $f_1, f_2, \ldots, f_n, \ldots$ converge towards $f$, in the sense of generalized functions. Then, in accordance with (1.8b) and (1.14a), for any basic function $\psi(\xi)$, we have

$$\lim_{n\to\infty} \langle f_n', \psi \rangle = \lim_{n\to\infty} \langle f_n, -\psi' \rangle = \langle f, -\psi' \rangle = \langle f', \psi \rangle,$$

i.e., /9

$$\lim_{n\to\infty} \langle f_n', \psi \rangle = \langle f', \psi \rangle \tag{1.15a}$$

or

$$\lim_{n\to\infty} f_n' = f' . \tag{1.15b}$$

Let us now examine the convolution of generalized functions. Let $f(\xi)$ and $g(\xi)$ be ordinary functions, absolutely integrable over the entire line. Then, their convolution is defined by the relationship

$$h(\xi) = f(\xi) * g(\xi) = \int_{-\infty}^{+\infty} f(\eta)\, g(\xi - \eta)\, d\eta , \tag{1.16}$$

in which h(ξ) also is an absolutely integrable function. The functional of , defined by function h(ξ), in accordance with (1.2), is written in the following form,

$$\langle h, \psi \rangle = \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{+\infty} f(\eta) g(\xi - \eta)\, d\eta \right] \psi(\xi)\, d\xi = \\ = \int_{-\infty}^{+\infty} f(\eta) \left[ \int_{-\infty}^{+\infty} g(\zeta) \psi(\eta + \zeta)\, d\zeta \right] d\eta .$$

Therefore, the convolution of two generalized functions f and g is defined by the relationship

$$\langle f * g, \psi \rangle = \langle f, \langle g, \psi(\xi + \eta) \rangle \rangle \tag{1.17a}$$

or, in symbolic form,

$$\langle f * g, \psi \rangle = \int_{-\infty}^{+\infty} f(\xi) \left[ \int_{-\infty}^{+\infty} g(\eta) \psi(\xi + \eta)\, d\eta \right] d\xi . \tag{1.17b}$$

It can be shown that the inner functional (1.17) is an infinitely differentiable function, but not finite and, consequently, it does not belong to space $\mathcal{K}_1$. Therefore, in the general case, the outer functional (1.17) is not defined. However, in sufficiently general assumptions, relationship (1.17), nevertheless, has meaning. We introduce only one condition, which is sufficient for the purposes of this work.

It is said that a generalized function (functional) f is concentrated in closed interval [a, b], if, for any basic function ψ(ξ), which reverts to zero in the open interval (a - ε, b + ε), for as small as desired ε > 0, there is an equality $\langle f, \psi \rangle = 0$. Such generalized functions (functionals) also are called finite. If regular functional (1.2), determined by ordinary function f(ξ), is concentrated in closed interval [a, b], this means that functions f(ξ) almost everywhere (with the exception of the set of zero measure points) outside [a, b] revert to zero.

For convolution of generalized functions f and g, there is the following proposition. For the truth of equality (1.17), it is sufficient that one of the generalized functions f, g be concentrated in a certain, bounded closed interval.

For some bounded operations, the convolution of generalized functions are continuous. For the generalized functions defined

In $\mathcal{K}_1$, there is the following simple, sufficient condition: /10
If the sequence of generalized functions $f_1, f_2, f_3, \ldots, f_n, \ldots$, concentrated in a bounded interval, converges toward a generalized function (obviously also concentrated in the same interval), for generalized function g, there is the equality

$$\lim_{n\to\infty} \langle f_n * g \rangle = \langle f * g \rangle$$

or, in symbolic form,

$$\lim_{n\to\infty} f_n * g = f * g \ .$$

In the preceding statement, it was assumed that basic functions $\psi(\xi)$ and functionals $\langle f, \psi \rangle$ take only real values. However, complex generalized functions can be defined in precisely the same way. For this, it is necessary to change from a space of real basic functions to the space of complex basic functions (i.e., infinitely differentiable and finite), which we, as before, will designate $\mathcal{K}_1$. In this case, each complex, locally integrable function $f(\xi)$ corresponds to functional

$$\langle f, \psi \rangle = \int_{-\infty}^{+\infty} f^*(\xi)\,\psi(\xi)\,d\xi \ , \tag{1.2a}$$

where the asterisk designates a complexly conjugate quantity. Relationships (1.6) and (1.7) take the form

$$\begin{aligned} \langle f_1 + f_2 , \psi \rangle &= \langle f_1, \psi \rangle + \langle f_2, \psi \rangle \ , \\ \langle \lambda f , \psi \rangle &= \lambda^* \langle f, \psi \rangle = \langle f, \lambda^* \psi \rangle \end{aligned} \tag{1.6a}$$

and

$$\langle a f, \psi \rangle = \langle f, a^* \psi \rangle \ , \tag{1.7a}$$

where $\alpha$ is a complex number and $a(x)$ is a complex, infinitely differentiable function. Each complex generalized function can be compared with the complexly conjugated generalized function $f^*$, by the formula

$$\langle f^*, \psi \rangle = \langle f , \psi^* \rangle^* \ . \tag{1.18}$$

The results obtained for real generalized functions, basically, are automatically carried over to the complex case, with allowance for changes flowing from relationships (1.2a), (1.6a), (1.7a) and (1.18).

We have given a brief survey of certain operations for generalized functions, defined by means of linear, continuous functionals in basic space $\mathcal{K}_1$. Before we proceed to subsequent material, we note that there are many other basic spaces, in which generalized functions can be defined in the same manner. In this case, the properties of generalized functions and definition of operations are basically preserved. The set of generalized functions determined in different basic spaces do not coincide, although they intersect. It turns out that, for solution of different problems, it is convenient to use different basic spaces.

We now proceed to examination of Fourier transforms of generalized functions. In this case, we will always understand $\mathcal{K}_1$ to be the space of the complex basic functions defined above. For any basic function $\psi(\xi)$, we examine the Fourier transform, with the complex parameter $\Omega = \lambda + i\tau$

$$\mathcal{F}[\psi(\xi)] = \tilde{\psi}(\Omega) = \int_{-\infty}^{+\infty} \psi(\xi)\, e^{i 2\pi \Omega \xi}\, d\xi \,. \tag{1.19}$$

Assuming here that $\tau = 0$, we obtain an ordinary Fourier transform $\tilde{\psi}(\lambda)$. Since $\psi(\xi)$ is a finite function (the integral actually is taken over a finite interval), in (1.19), differentiation over $\Omega$ can be carried out an unlimited number of times under the integral sign and, consequently, $\tilde{\psi}(\Omega)$ is an integral analytical function.

Derivatives $\psi(\xi)$ pertain to space $\mathcal{K}_1$. Therefore, by /11
partial integration, we have

$$\int_{-\infty}^{+\infty} \psi'(\xi)\, e^{i 2\pi \Omega \xi}\, d\xi = -\int_{-\infty}^{+\infty} (i 2\pi \Omega)\, \psi(\xi)\, e^{i 2\pi \Omega \xi}\, d\xi = -i 2\pi \Omega\, \tilde{\psi}(\Omega).$$

Continuing this process, we find that, for any real number q

$$\mathcal{F}[\psi^{(q)}(\xi)] = (-i 2\pi \Omega)^{q}\, \mathcal{F}[\psi(\xi)] \,. \tag{1.20}$$

Further, for any basic function $\psi(\xi)$, by virtue of its finite nature, such an $a > 0$ is found, that $\psi(\xi)$ reverts to 0, for $|\xi| \geq a$. Then, in accordance with (1.20), we have an estimate as $\Omega \to \infty$

$$|\Omega|^q |\tilde{\psi}(\Omega)| = \left| \int_{-a}^{a} \psi^{(q)}(\xi) e^{i 2\pi \Omega \xi} d\xi \right| \leq C_q e^{a|\tau|}, \qquad (1.21)$$

where constants $C_q$ and a depend on concrete function $\tilde{\psi}(\Omega)$. The resulting relationships show that the Fourier transform $\tilde{\psi}(\Omega)$ of each basic function $\psi(\xi)$, reverting to zero for $|\xi| \geq a$, is an integral analytical function, which satisfies inequality (1.21) to infinity.

The reverse statement can be proved in a similar manner: any integral analytical function $\tilde{\psi}(\Omega)$, having the properties specified, is the Fourier transform of a certain infinitely differentiable function $\psi(\xi)$, which reverts to zero at $|\xi| \geq a$, in which

$$\mathcal{F}^{-1}[\tilde{\psi}(\lambda)] = \psi(\xi) = \int_{-\infty}^{+\infty} \tilde{\psi}(\lambda) e^{-i 2\pi \lambda \xi} d\lambda . \qquad (1.22)$$

Thus, a Fourier transform, considering its singleness, establishes a mutually unambiguous correspondence between all functions of space $\mathcal{K}_1$ and the set $\mathcal{Z}_1$ of all integral analytical functions, which satisfy condition (1.21) to infinity, i.e., $\mathcal{Z}_1 = \tilde{\mathcal{K}}_1$ and $\mathcal{K}_1 = \tilde{\mathcal{Z}}_1$ .

The limiting transition can be determined in set $\mathcal{Z}_1$, considering that functions $\tilde{\psi}_n(\Omega)$ converge to $\tilde{\psi}(\Omega)$, if their forms $\psi_n(\xi)$ converge to form $\psi(\xi)$. This definition is equivalent to the following: sequence $\tilde{\psi}_1(\Omega)$, $\tilde{\psi}_2(\Omega)$, . . . , $\tilde{\psi}_n(\Omega)$, . . . converges to $\tilde{\psi}(\Omega)$ in $\mathcal{Z}_1$, if, for any real q, the following inequality is satisfied

$$|\Omega \tilde{\psi}_n(\Omega)| \leq C_q e^{a|\tau|}$$

with constant $C_q$ and a, not dependent on n, and $\tilde{\psi}_n(\lambda)$ tends uniformly toward $\tilde{\psi}(\lambda)$ in each finite interval. With such a definition, the convergence of set $\mathcal{Z}_1$ forms a basic space, in which, by means of linear, continuous functionals, generalized functions can also be determined.

With generalized functions in basic space $\mathcal{Z}_1$, operations can be carried out, similar to those introduced above, for generalized functions in $\mathcal{K}_1$ space. The regular functional has the same form (1.2a). The operations of summing and multiplication by number and the limiting transition contain nothing new. The operation of multiplication by function a ($\Omega$), formally defined by the same equality (1.7a), now becomes fulfillable for a narrower class of functions, which satisfy conditions of the type

$$|a(\Omega)| \leqslant C\, e^{b|\tau|}(1+|\Omega|)^{q} \tag{1.23}$$

for certain b, q and C. /12

The derivative of the generalized function $g' = dg/d\Omega$ is defined by the same formula (1.14). In this case, generalized functions in $\mathcal{Z}_1$ are not only infinitely differentiable, but analytical.

Just as for generalized functions in $\mathcal{K}_1$, the convolution of generalized functions in $\mathcal{Z}_1$ is defined by formula (1.17). The sufficient condition formulated above, in order for definition (1.17) to have meaning, remain true in this case. The convolution operation is continuous for generalized functions in $\mathcal{Z}_1$.

Since there exists a mutually unambiguous correspondence between basic spaces $\mathcal{K}_1$ and $\mathcal{Z}_1$, with preservation of the operations of convergence, summation and multiplication by a number, an analogous correspondence can be established between linear continuous functionals in these spaces, i.e., between generalized functions in $\mathcal{K}_1$ and $\mathcal{Z}_1$. This correspondence is established in such a way, that in functionals corresponding absolutely to integrable functions, it would be converted into a correspondence between a function and its classical Fourier transform.

Let $f(\xi)$ be any absolutely integrable function and $\tilde{f}(\lambda)$ its Fourier transform. Then, for any function $\psi(\xi)$ from $\mathcal{K}_1$ and its Fourier transform $\tilde{\psi}(\lambda)$ from $\mathcal{Z}_1$, there is a correlation [18]

$$\langle f, \psi \rangle = \int_{-\infty}^{+\infty} f^{*}(\xi)\,\psi(\xi)\,d\xi = \int_{-\infty}^{+\infty} \tilde{f}^{*}(\lambda)\,\tilde{\psi}(\lambda)\,d\lambda = \langle \tilde{f}, \tilde{\psi} \rangle, \tag{1.24}$$

which is called the Parseval equality. This correlation is taken for definition of generalized function $\tilde{f}$ in space $\mathcal{Z}_1$, for any given generalized function f in space $\mathcal{K}_1$. The generalized function (functional) $\tilde{f}$ is called the Fourier transform of generalized function (functional) f, and this fact is written in symbolic form

$$\tilde{f} = \mathfrak{F}[f].$$

For the Fourier transform of generalized functions, the usual differentiation formulas are preserved. They can be written in symbolic form

$$P\left(\frac{d}{d\Omega}\right)\mathcal{F}[f] = \mathcal{F}[P(i\xi)f] \,, \qquad (1.25)$$
$$\mathcal{F}\left[P\left(\frac{d}{d\xi}\right)f\right] = P(-i\Omega)\mathcal{F}[f] \,,$$

where $P(x)$ is a polynominal.

The inverse Fourier transform operator $\mathcal{F}^{-1}$ is determined and $\tilde{f}$ in $\mathcal{Z}_1$ is converted into f in $\mathcal{K}_1$, by the same formula (1.24) (read from right to left), so that

$$\langle \mathcal{F}^{-1}[\tilde{f}], \psi \rangle = \langle \tilde{f}, \mathcal{F}[\psi] \rangle \qquad (1.26a)$$

or, in symbolic form,

$$\mathcal{F}^{-1}[\mathcal{F}[f]] = f \,, \quad \mathcal{F}[\mathcal{F}^{-1}[\tilde{f}]] = \tilde{f} \,. \qquad (1.26b)$$

The fact is extensively used in analysis, that the Fourier transform convolutions of integrable functions $f(\xi)$ and $g(\xi)$ equal the product of the Fourier transforms $\tilde{f}(\lambda)$ and $\tilde{g}(\lambda)$ of these functions. Under certain conditions, this equality holds true for generalized functions.

We present here only the simplest sufficient conditions. If, of two generalized functions $f_1$ and $f_2$, defined in space $\mathcal{K}_1$, one is finite, i.e., it is concentrated in a bounded interval, there occurs the formula

$$\mathcal{F}[f_1 * f_2] = \tilde{f}_1\tilde{f}_2 \,. \qquad (1.27)$$

As an example illustrating the execution of a transformation, by means of generalizing functions and explaining the meaning of a generalized Fourier transform, we introduce one result which is useful for the future. We have /13

$$\int_{-A_n}^{A_n} e^{i2\pi\lambda\xi}\, d\xi = \frac{\sin 2\pi\lambda A_n}{\pi\lambda} \,. \qquad (1.28)$$

For $\lim A_n = \infty$, the right side of this equality, as it is easy to see, is a delta-form sequence, so that

$$\lim_{n\to\infty}\int_{-A_n}^{A_n} e^{i2\pi\lambda\xi}\,d\xi = \delta(\lambda) \tag{1.29a}$$

in the sense of generalized functions. It is easy to see that the left side becomes the generalized Fourier transform of unity i.e., functions $\mathbf{1}(\xi)$, for all $\xi$ taking the value of 1. In fact, in conformance with (1.2 ), taking the complex conjugated quantities in both parts of the equality, and multiplying by the basic function $\tilde{\psi}(\lambda)$ and integrating over $\lambda$, we obtain

$$\lim_{n\to\infty}\int_{-A_n}^{A_n}\left[\int_{-\infty}^{+\infty}\tilde{\psi}(\lambda)\,e^{-i2\pi\lambda\xi}d\lambda\right]d\xi = \lim_{n\to\infty}\int_{-\infty}^{+\infty}\frac{\sin 2\pi\lambda A_n}{\pi\lambda}\tilde{\psi}(\lambda)d\lambda,$$

or, proceeding to the limit and using $\delta$-function definition (1.5)

$$\langle \mathbf{1},\psi\rangle = \int_{-\infty}^{+\infty}\mathbf{1}(\xi)\,\psi(\xi)\,d\xi = \tilde{\psi}(0) = \langle\delta,\tilde{\psi}\rangle. \tag{1.29b}$$

Now, let $f(\xi)$ be any locally integrable function, growing as $|\xi| \to \infty$, no faster than a certain positive power of $|\xi|$. Such a function can always be represented in the form

$$f(\xi) = f_0(\xi)\,(1+\xi^2)^m \,, \tag{1.30}$$

where $f_0(\xi)$ is a function, absolutely integrable on the entire straight line $-\infty < \xi < +\infty$. Since $f_0(\xi)$ has a classical Fourier transform, the following limit exists

$$\lim_{n\to\infty}\int_{-A_n}^{A_n} f_0(\xi)\,e^{i2\pi\lambda\xi}\,d\xi = \tilde{f}_0(\lambda) \,. \tag{1.31}$$

Since

$$\left|\int_{-A_n}^{A_n} f_0(\xi)\,e^{i2\pi\lambda\xi}d\xi\right| \leqslant \int_{-\infty}^{+\infty}|f_0(\xi)|\,d\xi \,,$$

the moduli of all terms of the sequence in the left side of equality(1.31) are bounded by a fixed constant and, consequently, it also converges, in the sense of generalized functions. Applying the operation $(-d^2/d\lambda^2 + 1)^m$ to both sides of (1.31) and

$$\langle X, \psi \rangle = \int_{-\infty}^{+\infty} X^*(\xi)\,\psi(\xi)\,d\xi \; . \qquad (1.33)$$

If $X(\xi)$ is an ordinary, integrable random process, the integral on the right side has meaning, and functional (1.33) is called regular. The operations on the generalized random processes are determined, in the same manner, as for the determinate generalized functions, and they basically have the same properties. Thus, for example, for derivative $X'(\xi)$, we have

$$\langle X', \psi \rangle = \langle X, -\psi' \rangle \; . \qquad (1.34)$$

We note that, while the derivative of a common random process may not exist in any probabilistic sense, the derivative of a generalized random process always exists, and it is a generalized random process. Thus, in the set of generalized random processes, for example, random processes are included, with uncorrelated values, obtained by differentiation of the processes with uncorrelated increments.

The mean value (mathematical expectation) of the generalized random process $X(\xi)$ is called a functional

$$m_x(\psi) = \overline{\langle X, \psi \rangle} \; , \qquad (1.35)$$

if it is defined and continuous in $\mathcal{K}_1$.

The correlation functional of generalized random process $X(\xi)$ is called a bilinear functional

$$B_x(\psi_1, \psi_2) = \overline{\langle X, \psi_1 \rangle \langle X, \psi_2 \rangle^*} \qquad (1.36)$$

if it is defined in $\mathcal{K}_1$ and is continuous over each of the arguments $\psi_1(\xi)$ and $\psi_2(\xi)$. If $X(\xi)$ is a common integrable process, in accordance with (1.33)

$$B_x(\psi_1, \psi_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K_x^*(\xi_1, \xi_2)\,\psi_1(\xi_1)\,\psi_2^*(\xi_2)\,d\xi_1\,d\xi_2 = \qquad (1.37)$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K_x(\xi_2, \xi_1)\,\psi_1(\xi_1)\,\psi_2^*(\xi_2)\,d\xi_1\,d\xi_2 \; ,$$

since $$K_x(\xi_2, \xi_1) = K_x^*(\xi_1, \xi_2) \; .$$ /15 using the possibility of an unlimited differentiation of the sequence of generalized functions, we finally find that, in the generalized sense,

$$\lim_{n\to\infty}\int_{-A_n}^{A_n} f(\xi)\, e^{i2\pi\lambda\xi} d\xi = \left(-\frac{d^2}{d\lambda^2}+1\right)^m \tilde{f}_0(\lambda) \tag{1.32}$$

i.e., for the functions presented in the form of (1.30), there always exists a generalized Fourier transform. This result could have been obtained immediately, by applying formula (1.25) to (1.31).

With this, we end the brief survey of the properties of generalized functions and operations on them.

The advantages of the use of generalized functions is determined by the fact that the operations can be performed formally by known rules, when, in the ordinary sense, they do not exist. For interpretation of the final result, it is not expressed by normal functions; it is convenient to use the capability of representation of any generalized function, in the form of the limit of a sequence of normal functions, which converge, in the sense of generalized functions. /14

We note that all the relationships presented remain true, in the case, when all the functions depend on, not one variable, but on m independent variables. In this case, functions m of variables having the same properties as with one variable, must be used with basic spaces $\mathcal{K}_m$ and $\mathcal{Z}_m$. Natural changes must be incorporated into the formulas, assuming that

$$d\xi = d\xi_1 d\xi_2 \ldots d\xi_m, \quad d\lambda = d\lambda_1 d\lambda_2 \ldots d\lambda_m,$$
$$\frac{d}{d\xi} = \frac{\partial^m}{\partial\xi_1\partial\xi_2\ldots\partial\xi_m}, \quad \frac{d}{d\lambda} = \frac{\partial^m}{\partial\lambda_1\partial\lambda_2\ldots\partial\lambda_m},$$
$$i2\pi\lambda\xi = i2\pi(\lambda_1\xi_1+\lambda_2\xi_2+\ldots+\lambda_m\xi_m)$$

etc.

Just as in the case of determinate functions, generalized random processes (functions) can be defined in space $\mathcal{K}_1$. If each basic function $\psi(\xi)$ is compared to random quantity $\Phi(\psi)$, it is said that the random functional has been assigned. A continuous, linear random functional defines the correlated random process $X(\xi)$, which can be written in symbolic form

In the case of white noise $Y(\xi)$

$$B_Y(\psi_1, \psi_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} b_Y(\xi_1)\,\delta(\xi_1 - \xi_2)\,\psi_1(\xi_1)\,\psi_2^*(\xi_2)\,d\xi_1 d\xi_2 = \int_{-\infty}^{+\infty} b_Y(\xi_1)\,\psi_1(\xi_1)\,\psi_2^*(\xi_1)\,d\xi_1 . \tag{1.38}$$

Just as in the case of determinate functions, generalized random processes can be defined in other basic spaces.

## 2. Study of Connection Between Uniform (in the Broad Sense) Processes and White Noise

We will use the capital letters of the Latin alphabet X, Y, Z, etc., for designations of random processes (in the general case, of complex ones), and the corresponding lower case letters x, y, z, etc., for designations of their realizations. For time and frequency, we retain the designations t and f; in all other cases, we will designate the independent variables by the lower-case letters of the Greek alphabet $\xi$, $\lambda$, etc. We designate averaging over a set of realizations by a line above, and the operator for taking the mathematical expectation, by the letter $\mathcal{E}$. For the first two features of random process $X(\xi)$, we have, respectively,

$$m_x(\xi) = \overline{X(\xi)} = \mathcal{E}[x(\xi)] ,$$

$$B_x(\xi_1, \xi_2) = \overline{X(\xi_1)\,X^*(\xi_2)} = \mathcal{E}[x(\xi_1)\,x^*(\xi_2)] .$$

The complexly conjugate quantities are designated by the asterisk superscript in the last formula.

The correlation function and dispersion are are defined by the relationships

$$K_x(\xi_1, \xi_2) = B_x(\xi_1, \xi_2) - m_x(\xi_1)\,m_x^*(\xi_2) ,$$

$$\mathcal{D}_x(\xi) = K_x(\xi, \xi) .$$

Further, during the entire work, we will consider random processes to be centered, i.e., it is assumed that

$$m_x(\xi) = 0 , \quad K_x(\xi_1, \xi_2) = B_x(\xi_1, \xi_2) .$$

We now turn to certain results of the general theory of random processes [7, 8, 10, 14]. In order to emphasize that the independent variable does not mean time without fail, random process $X(\xi)$ with a zero mean, finite dispersion and a correlation function, dependent only on the difference of the arguments

$$m_x(\xi) \equiv 0, \quad \mathcal{D}_x(\xi) = \text{const} < \infty, \qquad (2.1)$$
$$K_x(\xi_1, \xi_2) = K_x(\xi_1 - \xi_2)$$

we will call uniform, in the broad meaning.

Subsequently, if it is not stipulated to the contrary, it /16 is assumed that a random process has a correlation function, which is continuous on line $\xi_2 = \xi_1$ (and, consequently everywhere). This condition is necessary and sufficient for continuity of the process, which is quadratic on the average, i.e., for the truth of

$$\lim_{h \to 0} \overline{|X(\xi - h) - X(\xi)|^2} = 0 . \qquad (2.2)$$

The correlation function of such random processes can always be represented in the form

$$K_x(\xi_1 - \xi_2) = \int_{-\infty}^{+\infty} e^{i 2\pi\lambda(\xi_1 - \xi_2)} d\mathcal{F}(\lambda), \qquad (2.3)$$

where function $\mathcal{F}(\lambda)$ is real, diminishing and bounded,

$$\mathcal{F}(+\infty) - \mathcal{F}(-\infty) = K_x(0) = \mathcal{D}_x ,$$

is called the spectral function of process $X(\xi)$. In turn, $\mathcal{F}(\lambda)$ is given by the relationship

$$\mathcal{F}(\lambda_2) - \mathcal{F}(\lambda_1) = \lim_{T \to \infty} \int_{-T}^{T} \frac{e^{-i 2\pi\lambda_2 \xi} - e^{-i 2\pi\lambda_1 \xi}}{-i 2\pi \xi} K_x(\xi) d\xi , \qquad (2.4)$$

where $\lambda_1$ and $\lambda_2$ are any two points of continuity of this function. If function $\mathcal{F}(\lambda)$ is absolutely continuous, i.e., it has a summable derivative (spectral density)

$$W_x(\lambda) = \frac{d\mathcal{F}}{d\lambda},$$

(2.3) and (2.4) change into the Wiener-Kintchine relationship

$$K_x(\eta) = \int_{-\infty}^{+\infty} e^{i2\pi\lambda\eta} W_x(\lambda)\, d\lambda,$$
$$W_x(\lambda) = \int_{-\infty}^{+\infty} e^{-i2\pi\lambda\eta} K_x(\eta)\, d\eta, \tag{2.5}$$

where $\eta = \xi_1 - \xi_2$. In cases when the spectral density is integrable according to Riemann

$$\int_{-\infty}^{+\infty} W_x(\lambda)\, d\lambda < \infty,$$

there also are ordinary Riemann integrals in relationships (2.5).

We require random processes $Z(\lambda)$, with zero mean and increments, which are uncorrelated in the nonintersecting intervals

$$m_z(\lambda) \equiv 0,$$
$$\overline{[Z(\lambda_2) - Z(\lambda_1)][Z^*(\lambda_4) - Z^*(\lambda_3)]} = 0, \quad \lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4. \tag{2.6}$$

It usually is assumed here that the increment has a finite dispersion

$$\overline{|Z(\lambda_2) - Z(\lambda_1)|^2} < \infty \tag{2.7}$$

All the properties of such processes connected with increments do not depend on supplements with an arbitrary ("constant") random value. For example, $Z(\lambda)$ can be replaced by $Z(\lambda) - Z(\lambda_0)$, i.e., the process, taking zero values at point $\lambda_0$ with probability 1, can be considered. If a determinate real function is defined

$$\mathcal{F}(\lambda) = \begin{cases} \overline{|Z(\lambda) - Z(\lambda_0)|^2} \text{ for } \lambda \geqslant \lambda_0, \\ -\overline{|Z(\lambda) - Z(\lambda_0)|^2} \text{ for } \lambda < \lambda_0, \end{cases} \tag{2.8}$$

in accordance with (2.6), for $\lambda_2 \geq \lambda_1$ /17

$$\overline{|Z(\lambda_2) - Z(\lambda_1)|^2} = \mathcal{F}(\lambda_2) - \mathcal{F}(\lambda_1), \tag{2.9}$$

i.e., $\mathcal{F}(\lambda)$ is a nondiminishing, bounded (see 2.7) function, so that, at each point $\lambda$, there exist bounds on the left $\mathcal{F}(\lambda-0)$ and on the right $\mathcal{F}(\lambda+0)$. Such functions, as is known, can have the largest countable set of discontinuity points of the first order. In accordance with (2.2), it follows from relationship (2.9) that, at each point of continuity $\mathcal{F}(\lambda)$ (i.e., almost everywhere), the process with uncorrelated increments $Z(\lambda)$ is continuous in the mean square.

Subsequently, we frequently will use the concept of convergence of random qualities in the mean square to random quantity S

$$\underset{n\to\infty}{l.i.m.}\ S_n = S, \tag{2.10}$$

if

$$\lim_{n\to\infty} \overline{|S_n - S|^2} = 0.$$

occurs. We introduce the following convenient sign of convergence. The sequence of random quantities $S_1$, $S_2$, $S_3$, . . . with finite dispersions, converge in the mean square to a certain random quantity S, when, and only when

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \overline{S_m S_n^*} = C, \quad |C| < \infty \tag{2.10a}$$

as $m \to \infty$ and $n \to \infty$ independently of each other. In this case, of course,

$$\lim_{n\to\infty} \overline{S_n} = \overline{S},$$
$$\lim_{n\to\infty} \overline{|S_n|^2} = C = \overline{|S|^2}. \tag{2.10b}$$

The following proposition also is useful for the future: If sequences $S_{11}$, $S_{12}$, $S_{13}$ . . . and $S_{21}$, $S_{22}$, $S_{23}$ . . . converge in the mean square to random quantities $S_1$ and $S_2$, respectively,

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \overline{S_{1m} S_{2n}^*} = \overline{S_1 S_2^*}. \tag{2.11}$$

We recall how the Riemann integral is determined from random function $X(\xi)$

$$J_1 = \int_a^b g(\xi)\, X(\xi)\, d\xi \tag{2.12a}$$

and the Riemann-Stieltjes integral, from random function $Y(\xi)$

$$J_2 = \int_a^b g(\xi)\, dY(\xi) \ , \tag{2.12b}$$

where (a, b) is a finite or an infinite interval and $g(\xi)$ is determinate function. Let there be

$$a = \xi_1 < \xi_2 < \ldots < \xi_j < \ldots < \xi_{n+1} = b$$

a certain partitioning of interval (a, b) and

$$S_{1n} = \sum_{j=1}^{n} g(\xi_j)\, X(\xi_j)\, (\xi_{j+1} - \xi_j) \ ,$$
$$S_{2n} = \sum_{j=1}^{n} g(\xi_j)\, [\, Y(\xi_{j+1}) - Y(\xi_j)\,]$$

the sums approximating integrals (2.12a) and (2.12b) respectively. If, as $n \to \infty$, /18

$$\max_j \, (\xi_{j+1} - \xi_j) \xrightarrow[n \to \infty]{} 0 \ , \quad j = 1, 2, \ldots, n$$

and integral sums $S_{1n}$ and $S_{2n}$ converge in the mean square to certain random quantities $J_1$ and $J_2$, independently of the specific partitioning of interval (a, b), it is said that integrals (2.10) and (2.11) exist in the mean square sense and equal these random quantities. Necessary and sufficient conditions for this, in accordance with (2.10a), consist of the existence of the Riemann integral

$$\overline{|J_1|^2} = \int_a^b \int_a^b g(\xi_1)\, g^*(\xi_2)\, K_x(\xi_1, \xi_2)\, d\xi_1\, d\xi_2 \tag{2.13}$$

and the Riemann-Stieltjes integral

$$\overline{|\mathcal{J}_2|^2} = \int_a^b\int_a^b g(\xi_1) g^*(\xi_2)\, d K_Y(\xi_1, \xi_2) \tag{2.14a}$$

of ordinary determinate functions. In the case when $Y(\xi)$ is a random process with uncorrelated increments, in accordance with (2.6) and (2.9), the latter condition can be represented in equivalent form

$$\overline{|\mathcal{J}_2|^2} = \int_a^b |g(\xi)|^2\, d\mathcal{F}(\xi) . \tag{2.14b}$$

If, as $a \to -\infty$, $b \to +\infty$, integrals (2.13), and (2.14) converge, the corresponding integrals of the random functions (2.12a) and (2.12b) also converge in the mean square sense, and the limiting random quantities $\mathcal{J}_1$ and $\mathcal{J}_2$ are called the values of these integrals in the mean square, over an infinite interval.

Stochastic integrals (2.12a) and (2.12b) defined in this manner have the properties of ordinary integrals. For example, if, beside satisfaction of conditions ensuring the existence of integral (2.14), it is required that function $g(\xi)$ have, in interval (a, b), integrable according to Riemann, derivative $g'(\xi)$, a formula for integration by parts occurs

$$\int_a^b g(\xi)\, dX(\xi) = g(b) X(b) - g(a) X(a) - \int_a^b g'(\xi) X(\xi)\, d\xi . \tag{2.15}$$

We now examine the stochastic integral

$$X(\xi) = \int_{-\infty}^{+\infty} g(\eta, \xi)\, d Z(\eta) , \tag{2.16}$$

where $Z(\eta)$ is a process with uncorrelated increments, having a bounded dispersion. If the integral on the right side, for each $\xi$, converges in the mean square, formula (2.16) defines random process $X(\xi)$, which, in turn, can be integrated. If $g(\eta, \xi)$ and $h(\xi)$ are continuous functions, satisfying the conditions

$$\begin{aligned} &\int_a^b \int_{-\infty}^{+\infty} |g(\eta, \xi)|^2\, d\xi\, d\mathcal{F}(\eta) < \infty , \\ &\int_a^b |h(\xi)|^2\, d\xi < \infty , \end{aligned} \tag{2.17}$$

where $d\mathcal{F}(\eta) = \overline{|dZ(\eta)|^2}$ in the repeated integral, the order of integration can be changed, i.e., the following relationship holds true /19

$$\int_a^b k(\xi)\left[\int_{-\infty}^{+\infty} g(\eta,\xi)\, dZ(\eta)\right] d\xi = \\ = \int_{-\infty}^{+\infty}\left[\int_a^b k(\xi)\, g(\eta,\xi)\, d\xi\right] dZ(\eta). \tag{2.18}$$

If, beside these conditions, there exists the improper integral

$$\int_{-\infty}^{+\infty} k(\xi)\, g(\eta,\xi)\, d\xi ,$$

(2.18) occurs for the infinite interval (a, b).

We now proceed to establishment of the connection between uniform processes and white noise. In accordance with general theory, for any uniform process $X(\xi)$, there exists such a process with uncorrelated increments $Z(\lambda)$, that, for each fixed $\xi$, there is a spectral representation

$$X(\xi) = \int_{-\infty}^{+\infty} e^{i2\pi\lambda\xi}\, dZ(\lambda) , \\ \overline{|Z(\lambda_2) - Z(\lambda_1)|^2} = \mathcal{F}(\lambda_2) - \mathcal{F}(\lambda_1) \text{ for } \lambda_2 > \lambda_1, \tag{2.19}$$

where the integral converges in the mean square, and $\mathcal{F}(\lambda)$ is a function of spectral representation (2.3) of the correlation function of the uniform process. In this case, process $Z(\lambda)$ can be defined by the formula

$$Z(\lambda_2) - Z(\lambda_1) = \underset{T\to\infty}{\mathrm{l.i.m.}} \int_{-T}^{T} \frac{e^{-i2\pi\lambda_2\xi} - e^{-i2\pi\lambda_1\xi}}{-i\,2\pi\xi} X(\xi)\, d\xi , \tag{2.20}$$

where $\lambda_1$ and $\lambda_2$ are any two points of continuity of function $\mathcal{F}(\lambda)$.

On the other hand, for any process $Z(\lambda)$ with uncorrelated increments, having a finite dispersion

$$\overline{|Z(\lambda_2) - Z(\lambda_1)|^2} = \mathcal{F}(\lambda_2) - \mathcal{F}(\lambda_1) < \infty$$

for any $\lambda_2 \geq \lambda_1$, in accordance with (2.14b), the integral on the right side of (2.19) converges in the mean square for each fixed $\xi$ and, consequently, it defines a uniform random process, the correlation function of which satisfies relationship (2.3).

In this manner, relationships (2.19) and (2.20) establish a correspondence between the uniform processes and processes with uncorrelated increments, defined at the beginning of this section.

We now use the generalized function apparatus presented in the preceding section. Let function $\psi(\xi)$ belong to basic space $\mathcal{K}_1$, i.e., be infinitely differentiable, and revert identically to zero outside a certain finite interval. Its Fourier transform

$$\widetilde{\psi}(\Omega) = \int_{-\infty}^{+\infty} \psi(\xi)\, e^{-i2\pi\lambda\xi}\, d\xi ,$$

where $\Omega = \lambda + i\sigma$ always exists, belongs to basic space $Z_1 = \widetilde{K}_1$ and, in accordance with (1.21), on the actual axis (for $\sigma = 0$), it satisfies the inequality

$$|\widetilde{\psi}(\lambda)| \leq \frac{C_n}{1 + |\lambda|^n} \tag{2.21}$$

for any $n > 0$.

For any function $\widetilde{\psi}(\lambda)$ /20

$$\int_{-\infty}^{+\infty} |\widetilde{\psi}(\lambda)|^2\, d\varphi < \infty ,$$

and, consequently (see (2.14b ), the integral

$$J = \int_{-\infty}^{+\infty} \widetilde{\psi}(\lambda)\, d\, Z(\lambda) \tag{2.22a}$$

converges in the mean square and defines a random, linear, continuous functional in basic space $Z_1 = \widetilde{K}_1$. Since, in this case, formula for partial integration (2.15) holds true and, in accordance with (2.21), $\widetilde{\psi}(-\infty) = \widetilde{\psi}(+\infty) = 0$, from (2.22a) and (1.14) we obtain

$$\begin{aligned} J &= -\int_{-\infty}^{+\infty} \widetilde{\psi}'(\lambda)\, Z(\lambda)\, d\lambda = \\ &= \langle Z(\lambda), -\widetilde{\psi}'(\lambda) \rangle = \langle Z'(\lambda), \widetilde{\psi}(\lambda) \rangle . \end{aligned} \tag{2.22b}$$

In this manner, the generalized function fixed by functional (2.22) is derivative $Z'(\lambda)$ of process with uncorrelated increments $Z(\lambda)$.

We now multiply both sides of equality (2.19) by $\psi(\xi)$ and the integrate over $\xi$. Considering that, in this case, the order of integration can be changed in the repeated integral (see 2.18) and, using (2.22), we obtain

$$
\begin{aligned}
\langle X(\xi), \psi(\xi) \rangle &= \int_a^b X(\xi)\psi(\xi)\,d\xi = \\
&= \int_a^b \psi(\xi)\Big[\int_{-\infty}^{+\infty} e^{i2\pi\lambda\xi}\,dZ(\lambda)\Big]\,d\xi = \\
&= \int_{-\infty}^{+\infty} \Big[\int_a^b \psi(\xi)\,e^{i2\pi\lambda\xi}\,d\xi\Big]\,dZ(\lambda) = \\
&= \int_{-\infty}^{+\infty} \tilde{\psi}(\lambda)\,dZ(\lambda) = \langle Z'(\lambda), \tilde{\psi}(\lambda) \rangle,
\end{aligned}
\tag{2.23}
$$

where (a, b) is an interval, outside of which finite function $\psi(\xi)$ reverts identically to zero. By comparison of this result with (1.24), we see that $X(\xi)$ and $Z'(\lambda)$ are Fourier transforms of each other, in the meaning of generalized functions, so that the following symbolic relationships can be written

$$X(\xi) = \int_{-\infty}^{+\infty} Z'(\lambda)\,e^{i2\pi\lambda\xi}\,d\lambda, \tag{2.24}$$

$$Z'(\lambda) = \int_{-\infty}^{+\infty} X(\xi)\,e^{-i2\pi\lambda\xi}\,d\xi. \tag{2.25}$$

With consideration of the singleness of the forward and reverse Fourier transforms, it follows from here that there exists a mutually unambiguous relationship between the generalized derivatives $Z'(\lambda)$ of the random processes with uncorrelated increments and the complex, uniform (in the broad sense) processes $X(\xi)$.

We dwell brieflyon certain properties of the generalized, random process $Z'(\lambda)$. For this, we examine the correlation functional

$$B_{z'}(\tilde{\psi}_1, \tilde{\psi}_2) = \overline{\langle Z', \tilde{\psi}_1 \rangle \langle Z', \tilde{\psi}_2 \rangle^*} \tag{2.26}$$

In accordance with (2.22),

$$\langle Z'(\lambda), \tilde{\psi}(\lambda) \rangle = \int_{-\infty}^{+\infty} Z'(\lambda)\tilde{\psi}(\lambda)\,d\lambda = \int_{-\infty}^{+\infty} \tilde{\psi}(\lambda)\,dZ(\lambda), \tag{2.27}$$

where the first equality is symbolic and the integral of $Z'(\lambda)$ /21 is understood in the sense of generalized functions. Considering that $Z(\lambda)$ is a process with uncorrelated increments and

$$\overline{|dZ(\lambda)|^2} = dF(\lambda),$$

from (2.27), we obtain

$$B_{Z'}(\tilde{\varphi}_1, \tilde{\varphi}_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{\varphi}_1(\lambda_1)\tilde{\varphi}_2^*(\lambda_2)K_{Z'}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 = \int_{-\infty}^{+\infty} \tilde{\varphi}_1(\lambda)\tilde{\varphi}_2^*(\lambda)\, dF(\lambda) = \overline{|J|^2}, \tag{2.28}$$

where the first equality, again, is symbolic. It can be shown that any function of two variables $\tilde{\psi}(\lambda_1, \lambda_2)$ from the basic functional space $\mathfrak{X}_2 = \tilde{\mathcal{K}}_2$ can be represented, as the limit of a sequence of linear combinations

$$\sum_{i,j} a_{ij}\tilde{\varphi}_i(\lambda_1)\tilde{\varphi}_j^*(\lambda_2),$$

where $\tilde{\psi}_i(\lambda)$ and $\tilde{\psi}_j(\lambda)$ belong to the basic space $\mathfrak{X}_1 = \tilde{\mathcal{K}}_1$ is a function of one variable; consequently, relationship (2.28) defines, in basic space $\mathfrak{X}_2 = \tilde{\mathcal{K}}_2$ a linear, continuous functional, which can be written symbolically in the form

$$< K_{Z'}(\lambda_1,\lambda_2), \tilde{\psi}(\lambda_1,\lambda_2) > = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K_{Z'}(\lambda_1,\lambda_2)\tilde{\psi}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2, \tag{2.29}$$

so that, according to general definition (1.4), correlation function $K_{Z'}(\lambda_1, \lambda_2)$ of process $Z'(\lambda)$ is a generalized function.

If the process with uncorrelated increments $Z(\lambda)$ is such that nondiminishing function $F(\lambda)$ has a derivative, which can be integrated in each finite interval $(\lambda_1, \lambda_2)$, i.e., for any $\lambda_1 < \lambda_2$, there exists the relationship

$$\overline{|Z(\lambda_2) - Z(\lambda_1)|^2} = F(\lambda_2) - F(\lambda_1) = \int_{\lambda_1}^{\lambda_2} F'(\lambda)d\lambda, \tag{2.30}$$

by transforming the last integral in (2.28), we obtain the symbolic equality

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \widetilde{\Psi}_1(\lambda_1)\widetilde{\Psi}_2^*(\lambda_2) K_{Z'}(\lambda_1,\lambda_2)\, d\lambda_1 d\lambda_2 = \tag{2.31}$$
$$= \int_{-\infty}^{+\infty} \widetilde{\Psi}_1(\lambda)\widetilde{\Psi}_2^*(\lambda)\, \mathcal{F}'(\lambda)\, d\lambda ,$$

from which it follows that, in the sense of generalized functions,

$$K_{Z'}(\lambda_1, \lambda_2) = \mathcal{F}'(\lambda_1)\, \delta(\lambda_2 - \lambda_1) , \tag{2.32}$$

where $\delta(\lambda)$ is the $\delta$-function. In this manner, by satisfaction of condition (2.30), process $Z'(\lambda)$ is white noise, as it was defined in the introduction.

In summary, we find that a mutually unambiguous correspondence exists between white noise and uniform (in the broad sense) processes, having spectral density $W(\lambda) = \mathcal{F}'(\lambda)$ .

The nondiminishing, bounded function $\mathcal{F}(\lambda)$ , which is the spectral function of the uniform (in the broad sense) process $X(\xi)$, as is known, can be expanded into the sum of three components

$$\mathcal{F}(\lambda) = \mathcal{F}_1(\lambda) + \mathcal{F}_2(\lambda) + \mathcal{F}_3(\lambda) , \tag{2.33}$$

where $\mathcal{F}_1(\lambda)$ is an absolutely continuous function, having derivative $\mathcal{F}_1'(\lambda)$ , integrable in the Lebesgue sense; $\mathcal{F}_2(\lambda)$ is a stepwise function, with jumps in a finite or countable number of points; $\mathcal{F}_3(\lambda)$ is a continuous function, distinct from constants, the derivative of which almost everywhere, i.e., with the exception of the set of zero measurement points, equals zero. /22

The presence of jumps in function $\mathcal{F}(\lambda)$ is closely connected with the properties of correlation function $K_x(\eta) = K_x(\xi_1 - \xi_2)$ of process $X(\xi)$

$$\lim_{M\to\infty} \frac{1}{M}\int_0^M K_x(\eta)\, d\eta = \mathcal{F}(+0) - \mathcal{F}(-0) , \tag{2.34}$$
$$\lim_{M\to\infty} \frac{1}{M}\int_0^M |K_x(\eta)|^2 d\eta = \sum_j [\mathcal{F}(\xi_j+0) - \mathcal{F}(\xi_j-0)]^2,$$

where summing is carried out over all points of the discontinuity $\xi_j$.

Since the singular component $\mathcal{F}_3(\lambda)$ does not have practical meaning, satisfaction of condition (2.30) actually means that the corresponding uniform, random process does not have discrete components.

We note that, when condition (2.30) is not satisfied, realization of process $Z'(\lambda)$ has a still more complicated and random nature, than in the case of white noise. In the next section, it will be shown that, by their ergodic properties, processes which do not satisfy condition (2.30), are not suitable for our p· ɔoses. More than that, certain limitations have to be imposed on the old derivative functions $\mathcal{F}(\lambda)$. Therefore, subsequently, we will always assume that condition (2.30) is satisfied, in which the integral exists, in the Riemann sense.

Up to now, in making all contributions, we have explicitly or implicitly used the limited nature of functions $\mathcal{F}(\lambda)$. However, this limitation is not required in proceeding to generalized functions. Actually, for a uniform (in the broad sense) generalized process given by the symbolic equality

$$X(\xi) = \int_{-\infty}^{+\infty} e^{i 2\pi\lambda\xi} Z'(\lambda)\, d\lambda ,$$

we have

$$K_x(\xi_1, \xi_2) = \int_{-\infty}^{+\infty} e^{i 2\pi\lambda(\xi_2 - \xi_1)} \mathcal{F}'(\lambda)\, d\lambda .$$

In the sense of generalized functions, the Fourier transform on the right side exists (see section 1), for any exponential growth function

$$\mathcal{F}'(\lambda) = \mathcal{F}_0(\lambda)\,(1 + \lambda^2)^m , \tag{2.35}$$

where m is an arbitrary real number and $\mathcal{F}_0(\lambda)$ is a function, integrable within infinite limits. It is evident that, in execution of (2.35), function $\mathcal{F}(\lambda)$ also is an exponential growth function and, consequently, all the relationships, beginning with (2.23), hold true for such functions. In particular, if, for $b = \text{const} > 0$,

$$\mathcal{F}(\lambda) = b\lambda ,$$

then,

$$K_x(\xi_1, \xi_2) = b\, \delta(\xi_2 - \xi_1)$$

and, consequently, $X(\xi)$ is a uniform (in the broad sense) white noise, with intensity b.

The considerations introduced permit the correspondence between white noise and uniform (in the broad sense) processes to be expanded, to the case when both processes are generalized.

## 3. Theory of Estimation of Intensity According to Individual Realizations

Now, let generalized random process Y(t) be a nonstationary /23 white noise, with correlation function

$$K_Y(t_1,t_2) = b_Y(t_1)\,\delta(t_1 - t_2)\,. \tag{3.1}$$

Then, its Fourier transform (in the sense of generalized functions) S(f) is a uniform (in the broad sense), random process, with correlation function $K_S(f_1 - f_2)$ and spectral density

$$W_S(t) = b_Y(t) = \int_{-\infty}^{+\infty} K_S(\sigma)\, e^{-i2\pi\sigma t}\, d\sigma\,. \tag{3.2}$$

Processes Y(t) and S(f) are connected by the symbolic relationships

$$Y(t) = \int_{-\infty}^{+\infty} S(f)\, e^{-i2\pi f t}\, df\,, \tag{3.3a}$$

$$S(f) = \int_{-\infty}^{+\infty} Y(t)\, e^{i2\pi f t}\, dt\,. \tag{3.3b}$$

The theory of obtaining estimates of spectral density of actual uniform processes according to individual realizations can be generalized to the case of complex processes. On the basis of (3.1) and (3.2), this permits an estimate of the intensity $b_Y(t)$ to be obtained by realization of process S(f). Let

$$S_F(f) = \begin{cases} S(f) & \textbf{for } |f| \le F/2\,, \\ 0 & \textbf{for } |f| > F/2 \end{cases} \tag{3.4}$$

be a truncated realization of process S(f), and

$$y_{SF}(t) = \int_{-\infty}^{+\infty} S_F(f)\, e^{-i2\pi f t}\, df = \int_{-F/2}^{F/2} S(f)\, e^{-i2\pi f t}\, df \tag{3.5}$$

is its Fourier transform. We form a function, analogous to a periodogram.

$$I_{SF}(t) = \frac{|y_{SF}(t)|^2}{F} \quad . \tag{3.6}$$

Then, using known relationships, the estimate $\phi_{SF}(t)$ of the intensity $b_Y(t)$ can be represented in the following form

$$\varphi_{SF}(t) = \int_{-\infty}^{+\infty} w_{\mathcal{L}}(t-\xi)\, I_{SF}(\xi)\, d\xi \ , \tag{3.7}$$

where the actual weight functions $w_{\mathcal{L}}(t)$ satisfy the conditions

$$w_{\mathcal{L}}(t) = w_{\mathcal{L}}(-t) \ , \quad \int_{-\infty}^{+\infty} w_{\mathcal{L}}(t)\, dt = 1 \tag{3.8}$$

and, moreover, they form a $\delta$-form sequence (see section I). In accordance with (1.13), with introduction of parameter function F

$$\mathcal{L} = \mathcal{L}(F) \ , \quad \lim_{F\to\infty} \mathcal{L}(F) = \infty$$

we reckon /24

$$\begin{aligned} &w_{\mathcal{L}}(t) = \mathcal{L}\, w(\mathcal{L} t) = \frac{1}{\Delta T}\, w\left(\frac{t}{\Delta T}\right) \ , \\ &\lim_{F\to\infty} \frac{F^{1/q}}{\mathcal{L}} = \lim_{F\to\infty} \Delta T\, F^{1/q} = \infty \quad \text{for} \quad q > 1 \ , \end{aligned} \tag{3.9}$$

where w(t) is any function with an integrable square

$$\int_{-\infty}^{+\infty} |w(t)|^2 dt < \infty \ , \tag{3.10}$$

which satisfies conditions (3.8). For such functions,

$$\begin{aligned} &\tilde{w}(f) = \lim_{\mu\to\infty} \int_{-\mu}^{\mu} w(t)\, e^{i2\pi f t}\, dt \ , \\ &w(t) = \lim_{\mu\to\infty} \int_{-\mu}^{\mu} \tilde{w}(f)\, e^{-i2\pi f t}\, df \ , \end{aligned} \tag{3.11}$$

where limits exist, at least on the average.

For $\tilde{w}_{\mathcal{L}}(f)$, in accordance with (3.9), we have

$$\tilde{w}_{\mathcal{L}}(f) = \tilde{w}(f/\mathcal{L}) = \tilde{w}(\Delta T f), \quad \tilde{w}_{\mathcal{L}}(f) = \tilde{w}_{\mathcal{L}}(-f), \quad \tilde{w}_{\mathcal{L}}(0) = 1 . \tag{3.12}$$

Additionally, we will assume function w(t) is such, that $\tilde{w}(f)$ is bounded on all axes and has a derivative at zero, although it is unilateral. In this case, there exists the highest number $\gamma > 1$, for which

$$\lim_{\Delta T \to 0} \frac{1 - \tilde{w}_{\mathcal{L}}(f)}{|\Delta T f|^{\gamma}} = \lim_{\Delta T \to 0} \frac{1 - \tilde{w}(\Delta T f)}{|\Delta T f|^{\gamma}} = A_{\gamma} , \tag{3.13a}$$

and, on all axes, $-\infty < f < +\infty$,

$$\frac{1 - \tilde{w}_{\mathcal{L}}(f)}{|\Delta T f|^{\gamma}} = \frac{1 - \tilde{w}(\Delta T f)}{|\Delta T f|^{\gamma}} < B_{\gamma} , \tag{3.13b}$$

where $A_{\gamma}$ and $B_{\gamma}$ are certain constants. If, for example, there exists $\tilde{w}''(0)$ and $\tilde{w}'(0) = 0$, then, $\gamma = 2$.

In the relationships presented above, $\mathcal{L}$ and $\Delta T$ are the effective bandwidth of functions $\tilde{w}_{\mathcal{L}}(f)$ and $w_{\mathcal{L}}(t)$ respectively. It is convenient, for example, to suppose

$$\mathcal{L} = C_1 \int_{-\infty}^{+\infty} \tilde{w}_{\mathcal{L}}^2(f)\,df , \quad (\tilde{w}_{\mathcal{L}}(0) = 1) , \quad \Delta T = C_2 \int_{-\infty}^{+\infty} w_{\mathcal{L}}^2(t)\,dt \Big/ w_{\mathcal{L}}^2(0) , \tag{3.14}$$

where the constants are given by the expressions

$$C_1 = 1 \Big/ \int_{-\infty}^{+\infty} \tilde{w}^2(f)\,df , \quad C_2 = w^2(0) \Big/ \int_{-\infty}^{+\infty} w^2(t)\,dt . \tag{3.15}$$

The definition of the function $w_{\mathcal{L}}(t)$ given here encompasses all cases encountered in practice. In particular, when

$$w(t) = \begin{cases} 1 & \text{for } |t| \leq 1/2 , \\ 0 & \text{for } |t| > 1/2 , \end{cases} \tag{3.16}$$

in accordance with (3.7) and (3.9), we obtain the estimate /25

$$\varphi_{SF}(t) = \frac{1}{\Delta T} \int_{t-\Delta T/2}^{t+\Delta T/2} I_{SF}(\xi)\, d\xi \,, \tag{3.17}$$

practical application of which, by use of digital computer, has been studied in detail by the authors, in work [17].

In order for the estimate $\phi_{SF}(t)$ to be asymptotically unbiased and consistent in process S(f) and, consequently, in white noise Y(t) corresponding to it, certain limitations must be imposed. First, uniform (in the broad sense) process S(f) should be ergodic relative to the first and second moments, i.e., the following relations should be satisfied

$$\begin{gathered} \lim_{F \to \infty} \mathcal{E}\left[ \left| \frac{1}{F} \int_{-F/2}^{F/2} S(f)\, df \right|^2 \right] = 0 \,, \\ \lim_{F \to \infty} \mathcal{E}\left[ \left| \frac{1}{F-|\sigma|} \int_{-F/2}^{F/2-|\sigma|} S(f)\, S^*(f+|\sigma|)\, df - K_s(|\sigma|) \right|^2 \right] = 0. \end{gathered} \tag{3.18}$$

A necessary and sufficient condition of ergodicity relative to the first moment is

$$\lim_{F \to \infty} \frac{1}{F} \int_0^F K_s(\sigma)\, d\sigma = 0 \,. \tag{3.19}$$

Concerning ergodicity relative to the correlation function, for Gaussian and certain other (for example, linear) processes, a necessary and sufficient condition is

$$\lim_{F \to \infty} \frac{1}{F} \int_0^F |K_s(\sigma)|^2\, d\sigma = 0 \,. \tag{3.20}$$

Since, in expansion of spectral function (2.33), we disregarded the singular component $\mathcal{F}_3$ for the processes indicated, in accordance with (2.34), a necessary and sufficient condition of ergodicity is absolute continuity of the spectral function or, which is the same thing, existence of a derivative, which is integrable in each finite interval (see 2.30). We recall that the derivative of the spectral function is spectral density $W_S(t)$ of process S(f) or intensity $b_Y(t)$ of white noise Y(t).

With the assumptions made, it is easy to show that, not only are conditions (3.19) and (3.20) satisfied, but

$$\lim_{f \to \infty} K_s(f) = 0$$

However, as $F \to \infty$, in order for the estimate $\phi_{SF}(t)$ to be asymptotically unbiased and consistent, an additional limitation must be imposed on the correlation function (or spectral density) of process S(f). We suppose that

$$\int_{-\infty}^{+\infty} | f^q K_s(f) | \, df < \infty \quad , \quad q > 0 \, . \qquad (3.21)$$

By the use of the reverse Fourier transform of the convolution, the expression for estimate (3.7) can be rewritten *in the form*

$$\varphi_{SF}(t) = \int_{-\infty}^{+\infty} \tilde{w}_L(f) \left(1 - \frac{|f|}{F}\right) r_{SF}(f) \, e^{-i2\pi f t} \, df \, , \qquad (3.22)$$

where /26

$$r_{SF}(\sigma) = \begin{cases} \int\limits_{-F/2}^{F/2-|\sigma|} S(f) S^*(f+|\sigma|) \, df & \text{for } F/2 \geq \sigma \geq 0 \, , \\ \int\limits_{-F/2}^{F/2-|\sigma|} S^*(f) S(f+|\sigma|) \, df & \text{for } -F/2 \leq \sigma < 0 \, , \end{cases} \qquad (3.23)$$

$$\left(1 - \frac{|f|}{F}\right) r_{SF}(f) = \int_{-\infty}^{+\infty} I_{SF}(t) \, e^{i2\pi f t} \, dt \, .$$

We examine the expression for shifting the estimate $B[\phi_{SF}(t)]$. In accordance with (3.2) and (3.22),

$$\begin{aligned} B[\varphi_{SF}(t)] &= \mathcal{E}[\varphi_{SF}(t)] - b_Y(t) = \\ &= -\int_{-\infty}^{+\infty} K_s(f) [1 - \tilde{w}_L(f)] \, e^{-i2\pi f t} \, df \; - \\ &\quad - \int_{-\infty}^{+\infty} \frac{|f|}{F} K_s(f) \, \tilde{w}_L(f) \, e^{-i2\pi f t} \, df \, . \end{aligned} \qquad (3.24)$$

We always can select such a w(t), that condition

$$\gamma \geq q$$

is satisfied (see 3.13, 3.21). Then, on consideration that

$$|\tilde{w}_{\ell}(f)| < M$$

on all axes, and with the **use** of (3.13) and (3.21), as $F \to \infty$, we obtain the asymptotic relationships

$$\int_{-\infty}^{+\infty} K_s(f)[1 - \tilde{w}_{\ell}(f)]\, e^{-i2\pi f t}\, df \sim (\Delta T)^q\, \Phi_1(t)\,,$$
$$\int_{-\infty}^{+\infty} \frac{|f|}{F} K_s(f)\, \tilde{w}_{\ell}(f)\, e^{-i2\pi f t}\, df \sim \frac{(\Delta T)^q}{(\Delta T)^q F}\, \Phi_2(t)\,,$$

where $\Phi_1(t)$ and $\Phi_2(t)$ are continuous functions. Since, in accordance with (3.9),

$$(\Delta T)^q F \xrightarrow[F \to \infty]{} \infty\,,$$

we finally find that, for a sufficiently large F,

$$B[\varphi_{sF}(t)] \sim (\Delta T)^q\, \Phi_1(t)\,. \tag{3.25}$$

With the assumptions made, for dispersion of the estimate $\mathcal{D}[\varphi_{sF}(t)]$, there is an asymptotic formula [22]

$$\mathcal{D}[\varphi_{sF}(t)] \sim \frac{c\, b_Y^2(t) \int_{-\infty}^{+\infty} |\tilde{w}_{\ell}(t)|^2\, dt}{\Delta T\, F}\,, \tag{3.26}$$

where c is a constant. The change in $\Delta T$ should be so matched with change in F, that the square of the displacement and the dispersion of the estimate change at the same rate. In accordance with (3.25) and (3.26), in this case, /27

$$\Delta T = O\left(F^{-\frac{1}{1+2q}}\right), \quad \mathcal{D}[\varphi_{sF}] = O\left(F^{-\frac{2q}{1+2q}}\right). \tag{3.27}$$

It is seen from these relationships that, for $q < 1$, convergence of the estimates towards the intensity is very slow and, for $q > 2$, further increase in q slightly affects the convergence rate. In such a manner that the process could be actually analyzed, the absolute value of correlation function $|K_S(\sigma)|$ should decrease more rapidly than $1/\sigma^{2+\varepsilon}$ (see 3.21) and, consequently, the spectral density $W_S(t)$ of process $S(f)$ (or intensity $b_Y(t)$ of white noise $Y(t)$) should have at least one continuous derivative.

We call estimates considered estimates of the first kind. Together with them, estimates of the second kind are widespread. They frequently are called "instrumental," since, in form, they are a mathematical recording of an electronic spectrometer line diagram. Let $h_{\Delta T}(f,t_0)$ be such an actual function of f ($t_0$ is a parameter), that the module of its Fourier transform

$$\tilde{h}_{\Delta T}(t,t_0) = \int_{-\infty}^{+\infty} h_{\Delta T}(f,t_0)\, e^{-i2\pi f t}\, df \tag{3.28}$$

is an even function

$$|\tilde{h}_{\Delta T}(t,t_0)| = |\tilde{h}_{\Delta T}(-t,t_0)|$$

and it differs appreciably from zero, only in a band with effective width $\Delta T$ around the value of t, equal to $t_0$ and $-t_0$. Then, an estimate of the second kind of white noise intensity is written in the form

$$\varphi_{S\delta T}(t) = \int_{-\infty}^{+\infty} h_{\delta T}(f,0)\, |S_{\Delta T}(f,t)|^2\, df\,, \tag{3.29}$$

where

$$S_{\Delta T}(f,t_0) = \int_{-\infty}^{+\infty} h_{\Delta T}(\eta,t_0)\, S(f-\eta)\, d\eta = \int_{-\infty}^{+\infty} h_{\Delta T}(f-\eta,t_0) S(\eta)\, d\eta. \tag{3.30}$$

Here, functions $h_{\Delta T}(f, t_0)$ and $h_{ST}(f, 0)$ can be interpreted as transient pulse functions of a narrow band filter and a low frequency filter. With the use of (3.29) and (3.30), for calculation of the estimate, the requirement of physical realizability of the filters is not obligatory. As $\delta T \to 0$, in order for the estimate of the white noise intensity $\phi_{S\delta T}(t)$ to be asymptotically unbiased and consistent, a connection should be established between $\delta T$ and $\Delta T$, the nature of which is determined by the same considerations, as for estimates of the first kind. In particular, the following relationship should be satisfied

$$\lim_{\delta T \to 0} \frac{\Delta T}{\delta T} = \infty \qquad (3.31)$$

Formulas (3.7) and (3.29) permit determination of the estimate of the white noise intensity Y(t), by carrying out a series of operations, to realize uniform (in a broad sense) process S(f), corresponding to it. However, if the white noise is fixed, it is impossible to calculate realization S(f) directly, by means of a Fourier transform, since realization y(t) is a generalized function, and its value is not definite at a single point. In this situation, in order to give formulas (3.7) and (3.29) actual meaning, the fact must be used that any generalized function can be obtained, as the limit of a sequence of ordinary /28 functions, if the convergence is understood in the sense of generalized functions (see section 1).

If $Y_n(t)$ is a random function with finite dispersion, and $\psi(t)$ is any infinitely differentiable function, which reverts to zero outside a certain finite interval, then, integral

$$\langle Y_n, \psi \rangle = \int_{-\infty}^{+\infty} \psi(t) Y_n(t)\, dt \qquad (3.32)$$

always exists, in the mean square sense, and it defines a linear, continuous, random functional, in basic functional space $\mathcal{K}_1$. Let $Y_1(t)$, $Y_2(t)$, $Y_3(t)$, . . . be a sequence of ordinary random processes with finite dispersion, which converges, in the sense of generalized functions, to white noise Y(t), with correlation function

$$K_Y(t_1, t_2) = \ell_Y(t_1)\, \delta(t_1 - t_2) \, .$$

This means that, for each fixed function $\psi_1(t)$ from $\mathcal{K}_1$, the sequence of random quantities

$$\langle Y_1, \psi_1 \rangle, \langle Y_2, \psi_1 \rangle, \langle Y_3, \psi_1 \rangle, \ldots \qquad (3.33)$$

converges, in the mean square sense, towards the value $\langle Y, \psi_1 \rangle$ of a random functional, which fixes the white noise, i.e.,

$$\underset{n \to \infty}{\mathrm{l.i.m.}} \langle Y_n, \psi_1 \rangle = \langle Y, \psi_1 \rangle \, . \qquad (3.34)$$

In accordance with (2.11), it follows from here that

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \overline{\langle Y_m, \psi_1 \rangle \langle Y_n, \psi_2 \rangle^*} = \lim_{\substack{m\to\infty\\ n\to\infty}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi_1(t_1)\psi_2^*(t_2)\overline{Y_m(t_1)Y_n^*(t_2)}\,dt_1 dt_2 = \tag{3.35}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi_1(t_1)\psi_2^*(t_2) b_Y(t_1)\delta(t_1 - t_2)\,dt_1 dt_2 = \int_{-\infty}^{+\infty} \psi_1(t_1)\psi_2^*(t_1) b_Y(t_1)\,dt_1 ,$$

where $\psi_1(t)$ and $\psi_2(t)$ are any two functions from $\mathcal{K}_1$. Reasoning further, the same as in the change from (2.28) to (2.29), from (3.35) we obtain

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi(t_1,t_2) Y_m(t_1) Y_n^*(t_2)\,dt_1 dt_2 =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi(t_1,t_2) b_Y(t_1)\delta(t_1 - t_2)\,dt_1 dt_2 ,$$

where $\psi(t_1, t_2)$ is any function from basic space $\mathcal{K}_2$. This is equivalent to the fact (see section 1) that, as $m \to \infty$ and $n \to \infty$ independently of each other,

$$\overline{Y_m(t_1)Y_n^*(t_2)} \longrightarrow K_Y(t_1,t_2) = b_Y(t_1)\delta(t_1 - t_2) \tag{3.37}$$

in the sense of convergence of generalized functions. Conversely, if (3.37) takes place, and that means (3.36) a second equality (3.35) follows from this, since functions $\psi_1(t_1)\psi_2^*(t_2)$ belong to space $\mathcal{K}_2$. Then, the first equality (3.35) shows that a limit exists

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \overline{\langle Y_m, \psi_1 \rangle \langle Y_n, \psi_1 \rangle^*}$$

and, consequently, in accordance with (2.10), in the mean square sense, sequence of random quantities (3.33) converges. In this manner, fulfillment of relationship (3.37) is a necessary and sufficient condition for convergence of sequence of random processes $Y_1(t)$, $Y_2(t)$, $Y_3(t)$, . . . , with finite dispersion, to white noise $Y(t)$, in the sense of generalized functions. Since the Fourier transform does not disrupt the convergence, from here there also follows convergence of sequence $S_1(f)$, $S_2(f)$, $S_3(f)$, . . . , where /29

$$S_n(f) = \int_{-\infty}^{+\infty} Y_n(t)\, e^{i2\pi f t}\,dt , \tag{3.38}$$

towards stationary process $S(f)$, corresponding to white noise $Y(t)$. It is seen from the results obtained that an infinite set of sequences of common random processes can be constructed which converge in the sense indicated, towards one and the same white noise.

For simplicity, we assume that white noise Y(t) and all $Y_1(t)$, $Y_2(t)$, etc., differ from zero, only in a finite interval (a, b). We introduce the designation

$$Y_{nsF}(t) = \int_{-F/2}^{F/2} S_n(f) e^{-i2\pi f t} df = \int_{-\infty}^{+\infty} S_n(f) \Phi_F(f) e^{-i2\pi f t} df, \tag{3.39}$$

where

$$\Phi_F(f) = \begin{cases} 1 & \text{for } |f| \leq F/2, \\ 0 & \text{for } |f| > F/2. \end{cases}$$

Then

$$Y_{nsF}(t) = \int_{-F/2}^{F/2} \left[ \int_a^b Y_n(\xi) e^{i2\pi f \xi} d\xi \right] e^{-i2\pi f t} df = \int_a^b Y_n(\xi) \frac{\sin \pi F(t-\xi)}{\pi(t-\xi)} d\xi \tag{3.40}$$

since, in this case, the integration order can be changed. With consideration of continuity of the convolution (see section I), it follows from (3.40) and convergence $Y_n(t) \to Y(t)$, that

$$Y_{nsF}(t) \xrightarrow[n\to\infty]{} \int_a^b Y(\xi) \frac{\sin \pi F(t-\xi)}{\pi(t-\xi)} d\xi$$

in the sense of generalized functions. Further, by the use of the reverse Fourier transform of the convolution of the generalized functions (see section I) and considering that

$$\int_{-\infty}^{+\infty} \frac{\sin \pi F \xi}{\pi \xi} e^{i2\pi f \xi} d\xi = \Phi_F(f),$$

we transform the latter relationship to

$$Y_{nsF}(t) \xrightarrow[n\to\infty]{} \int_{-F/2}^{F/2} S(f) e^{-i2\pi f t} df = Y_{sF}(t).$$

Since a common random function is on the right, convergence actually takes place in the ordinary sense. By repeating the reasoning presented, but for functions of two variables, by means of (3.40), we find that, from convergence

$$Y_m(t_1)Y_n(t_2) \longrightarrow b_Y(t_1)\delta(t_1 - t_2)$$

there follows /30

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \overline{Y_{mSF}(t)Y^*_{nSF}(t)} = \int_a^b b_Y(\xi)\left[\frac{\sin \pi F(t-\xi)}{\pi(t-\xi)}\right]^2 d\xi .$$

In this manner, in accordance with (2.10), we finally obtain

$$\underset{n\to\infty}{l.i.m.}\, Y_{nSF}(t) = \int_{-F/2}^{F/2} S(f) e^{-i2\pi f t} df = Y_{SF}(t) . \qquad (3.41)$$

We designate the result of application of operator (3.7) to realization $y_n(t)$ of random process $Y_n(t)$ by $\phi_{nSF}(t)$. Then, with (3.39) and (3.40) taken into account, we obtain

$$\begin{aligned}\varphi_{nSF}(t) &= \frac{1}{F}\int_{-\infty}^{+\infty} w_L(t-\xi)\left|\int_{-F/2}^{F/2} s_n(f) e^{-i2\pi f\xi} df\right|^2 d\xi = \\ &= \frac{1}{F}\int_{-\infty}^{+\infty} w_L(t-\xi)\left|\int_a^b y_n(\eta)\frac{\sin \pi F(\xi-\eta)}{\pi(\xi-\eta)} d\eta\right|^2 d\xi .\end{aligned} \qquad (3.42)$$

It follows from (3.41) that, as $n \to \infty$, the finite-dimensional distributions of probabilities of processes $Y_{nSF}(t)$ converge to finite-dimensional distributions of process $Y_{SF}(t)$. The question arises: under what condition does convergence of distribution function $\mathcal{F}_n(x)$ of estimate $\phi_{nSF}$ to distribution function $\mathcal{F}(x)$ of estimate $\phi_{SF}$, for each fixed t, flow from this? We note that, from convergence of the distributions at each point of continuity of the limiting function, convergence of the moments still does follow. In our case, the limiting estimate $\phi_{SF}$ has the first two finite moments and, in order for

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty} x\, d\mathcal{F}_n(x) = \mathcal{E}[\varphi_{SF}] ; \quad \lim_{n\to\infty}\int_{-\infty}^{+\infty} x^2\, d\mathcal{F}_n(x) = \mathcal{E}[|\varphi_{SF}|^2] ,$$

permissibility of transition to the limit under the integral sign is required.

Now, we will suppose that all processes $Y_n(t)$ have continuous realizations $y_n(t)$, with probability 1. This case is of the greatest interest to us, since, for example, the majority of processes which describe oscillations of physical or mechanical systems have such a property. Having substituted the square of the module, in the form of the product of the complexly conjugate quantities, in formula (3.42) and by changing the order of integration, we see [13] that, for fixed t, $\phi_{nSF}$ is a continuous quadratic functional in the space of all functions, continuous in (a, b). Then, in accordance with the known theorem [7, 8], for convergence of distribution of estimates $\phi_{nSF}$ to distribution $\phi_{SF}$, with fixed t, it is sufficient that processes $Y_n(t)$, for all n and $t_1$, $t_2$, satisfy the condition

$$\mathcal{E}[|y_n(t_1) - y_n(t_2)|^{\alpha}] \leqslant H|t_1 - t_2|^{1+\beta} \tag{3.43}$$

where $\alpha$, $\beta$ and H are certain positive constants. In order to more realistically represent the nature of these limitations, we note that condition (3.43) is fulfilled, for example, for $\alpha = 2$ and $\beta = 1$, when correlation functions $K_{Y_n}(t_1, t_2)$ of processes /31
$Y_n(t)$, on line $t_1 = t_2$ and, consequently, everywhere, have the bounded, composite derivative

$$\frac{\partial K_{Y_n}(t_1, t_2)}{\partial t_1 \partial t_2} < C \ , \tag{3.43a}$$

i.e., for processes, which are differentiable in the mean square sense.

Of course, condition (3.43) is not necessary. For example, the model of the random process, which we use in the succeeding sections, does not satisfy this condition. In the partial case of a determinate amplitude, we give this process with the equation (see 4.2)

$$Y_n(t) = \sqrt{\mathfrak{D}_n}\,\sqrt{b_Y(t)}\,\cos G(t) \ , \tag{3.44}$$

where G(t) is a real, uniform, Markov process, $b_Y(t)$ is a real determinate function different from zero only in the interval (a, b), $\mathfrak{D}_n$ is a positive parameter dependent on n, in which

$$\lim_{n\to\infty} \mathcal{D}_n = \infty \ , \quad \lim_{\substack{m\to\infty \\ n\to\infty}} \frac{\mathcal{D}_m}{\mathcal{D}_n} = 1 \ . \tag{3.45}$$

Calculation of the second and fourth moments by the use of the probability of transition (4.6) gives

$$\begin{aligned}
&\overline{Y_n(t_1)\,Y_n(t_2)} = K_{Y_n}(t_1,t_2) = \sqrt{b_Y(t_1)b_Y(t_2)}\ \frac{\mathcal{D}_n}{2}\, e^{-\mathcal{D}_n|t_1-t_2|} \ , \\
&\overline{Y_m(t_1)Y_n(t_2)} = K_{Y_mY_n}(t_1,t_2) = \sqrt{b_Y(t_1)\,b_Y(t_2)}\ \frac{\sqrt{\mathcal{D}_m\mathcal{D}_n}}{2}\, e^{-\sqrt{\mathcal{D}_m\mathcal{D}_n}\,|t_1-t_2|} \ , \\
&\overline{Y_n(t_1)Y_n(t_2)Y_n(t_3)Y_n(t_4)} = \sqrt{b_Y(t_1)b_Y(t_2)b_Y(t_3)b_Y(t_4)}\ \frac{\mathcal{D}_n^2}{4}\ \cdot \\
&\qquad \cdot\left\{1+\frac{1}{2}e^{-2\mathcal{D}_n|t_2-t_3|}\right\} e^{-\mathcal{D}_n(|t_1-t_2|+|t_3-t_4|)} \ .
\end{aligned} \tag{3.46}$$

It is easy to see that $Y_1(t)$, $Y_2(t)$, $Y_3(t)$, . . . form a sequence which converges to white noise, with intensity $b_Y(t)$, since, as $n \to \infty$ ($D_n \to \infty$)

$$\frac{\mathcal{D}_n}{2}\, e^{-\mathcal{D}_n|\tau|} \longrightarrow \delta(\tau) \tag{3.47}$$

and, consequently,

$$K_{Y_mY_n}(t_1,t_2) \longrightarrow b_Y(t_1)\delta(t_1-t_2) \ , \quad K_{Y_n}(t_1,t_2) \longrightarrow b_Y(t_1)\,\delta(t_1-t_2) \ . \tag{3.48}$$

For fixed t, we calculate the mathematical expectation and second moment of the estimate $\phi_{nSF}(t)$. From (3.42), we obtain

$$\begin{aligned}
\mathcal{E}[\varphi_{nSF}(t)] = \frac{1}{F}\int_{-\infty}^{+\infty} w_\ell(t-\xi)\Big[\int_a^b\!\!\int_a^b K_{Y_n}(\eta_1,\eta_2)\ \cdot \\
\cdot\ \frac{\sin\pi F(\xi-\eta_1)}{\pi(\xi-\eta_1)}\ \frac{\sin\pi F(\xi-\eta_2)}{\pi(\xi-\eta_2)}\ d\eta_1 d\eta_2\Big]\, d\xi
\end{aligned} \tag{3.49}$$

and /32

$$\mathcal{E}[|\varphi_{nsF}(t)|^2] = \frac{1}{F^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w_\ell(t-\xi_1)\, w_\ell(t-\xi_2)\cdot$$

$$\cdot\left\{\int_a^\ell\int_a^\ell\int_a^\ell\int_a^\ell \mathcal{E}[y_n(\eta_1)y_n(\eta_2)y_n(\eta_3)y_n(\eta_4)]\,\frac{\sin\pi F(\xi_1-\eta_1)}{\pi(\xi_1-\eta_1)}\,\frac{\sin\pi F(\xi_1-\eta_2)}{\pi(\xi_1-\eta_2)}\cdot\right. \tag{3.50}$$

$$\left.\cdot\,\frac{\sin\pi F(\xi_2-\eta_3)}{\pi(\xi_2-\eta_3)}\,\frac{\sin\pi F(\xi_2-\eta_4)}{\pi(\xi_2-\eta_4)}\,d\eta_1 d\eta_2 d\eta_3 d\eta_4\right\} d\xi_1 d\xi_2 \ .$$

From the properties of the functions included in these formulas, it follows that, with the assumptions made earlier, one can only proceed to the limit as $n \to \infty$, under the signs of the first integrals in these formulas. We substitute the expression for $K_{Yn}(\eta_1, \eta_2)$ from (3.46) in (3.49). Proceeding to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \mathcal{E}[\varphi_{nsF}(t)] = \int_{-\infty}^{+\infty} w_\ell(t-\xi)\left[\int_a^\ell b_Y(\eta_1)\cdot\right. \tag{3.51a}$$

$$\left.\cdot\,\frac{\sin^2\pi F(\xi-\eta_1)}{\pi^2 F(\xi-\eta_1)^2}\,d\eta_1\right] d\xi = \mathcal{E}[\varphi_{sF}(t)] \ .$$

Since the integral stands inside, with a Fejer kernel [18], considering the properties of function $w_\ell(t)$ as $F \to \infty$, we finally obtain

$$\lim_{F\to\infty}\lim_{n\to\infty} \mathcal{E}[\varphi_{nsF}(t)] = b_Y(t) \ . \tag{3.52a}$$

In a similar manner, by substitution of the expression for the fourth moment from (3.46) in (3.50), we obtain

$$\lim_{n\to\infty} \mathcal{E}[|\varphi_{nsF}|^2] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w_\ell(t-\xi_1)\, w_\ell(t-\xi_2)\cdot$$

$$\cdot\left\{\left[\int_a^\ell b_Y(\eta_1)\,\frac{\sin^2\pi F(\xi_1-\eta_1)}{\pi^2 F(\xi_1-\eta_1)^2}\,d\eta_1\right]\cdot\right. \tag{3.51b}$$

$$\left.\cdot\left[\int_a^\ell b_Y(\eta_4)\,\frac{\sin^2\pi F(\xi_2-\eta_4)}{\pi^2 F(\xi_2-\eta_4)^2}\,d\eta_4\right]\right\} d\xi_1 d\xi_2 = \mathcal{E}[|\varphi_{sF}|^2]$$

and, correspondingly,

$$\lim_{F\to\infty}\lim_{n\to\infty}\mathcal{E}[|\varphi_{nSF}(t)|^2] = b_Y^2(t) \tag{3.52b}$$

or, considering (3.52a)

$$\lim_{F\to\infty}\lim_{n\to\infty}\mathcal{D}[\varphi_{nSF}(t)] = 0 \ . \tag{3.52c}$$

It is seen from the first relationship of (3.46) that, on the line $t_2 = t_1$, correlation function $K_{Y_n}(t_1, t_2)$ does not have a mixed derivative. In this manner, in the example being considered, despite the fact that condition (3.43a) is not fulfilled, not only does the probability distribution of estimates $\phi_{nSF}$ converge to the distribution of estimate $\phi_{SF}$, but the first two moments, at least (see 3.51). Estimate $\phi_{nSF}(t)$ of the white noise intensity is asymptotically unbiased and consistent in this case (see 3.52). /33

In all the formulas presented, the order of transition to the limit over n and F is significant. This holds true, not only for the example, but in the general case. It is seen from formulas (3.49) and (3.50) that, if initially $F \to \infty$, both limits revert to zero.

In this manner, for $\phi_{nSF}(t)$ to be a sufficiently effective estimate of intensity, the following condition must be fulfilled

$$\mathcal{D}_n \gg F, \quad \mathcal{L} = \frac{1}{\Delta T} \gg F \ . \ \frac{1}{\mathcal{L}} = \Delta T \ll T_x \ , \tag{3.53}$$

where, in the general case, $1/\mathcal{D}_n$ is on the order of the effective length of the correlation process $Y_n(t)$ and can be estimated, for example, by the formula

$$\frac{1}{\mathcal{D}_n} \sim \frac{\int_a^b\int_a^b K_{Y_n}(t_1,t_2)\,dt_1\,dt_2}{(b-a)\int_a^b K_{Y_n}(t,t)\,dt} \ . \tag{3.54}$$

In the frequency region corresponding to $Y_n(t)$, random process $S_n(f)$, in interval $|f| < \mathcal{D}_n$, is sufficiently close to uniform by properties, in the broad sense of the process.

The resulting estimates $\phi_{nSF}(t)$ of white noise intensity permit generalization. In fact, if, in determination of $Y_{nSF}(t)$ (see 3.39), in place of $\Phi_F(f)$, use is made of any function

$$\mathcal{V}_F(f) = \int_{-\infty}^{+\infty} h_0(t)\,e^{i2\pi ft}\,dt \ , \tag{3.55}$$

where $h_0(t)$ is the transition pulse function of the low frequency filter (not obligatorily physically realizable), permitting change in the order of integration in the relationship

$$Y_{nSF}(t) = \int_{-\infty}^{+\infty} v_F(f) \left[ \int_a^b Y_n(\xi) e^{i2\pi f \xi} d\xi \right] e^{-i2\pi f t} df = \int_a^b Y_n(\xi) h_0(t-\xi) d\xi , \qquad (3.56)$$

the entire subsequent course of reasoning remains in force, and we obtain, for the estimate $\phi_{nSF}(t)$, the formula

$$\varphi_{nSF}(t) = \frac{1}{F} \int_{-\infty}^{+\infty} w_L(t-\xi) \left| \int_{-\infty}^{+\infty} S_n(f) v_F(f) e^{-i2\pi f \xi} df \right|^2 d\xi = \frac{1}{F} \int_{-\infty}^{+\infty} w_L(t-\xi) \left| \int_a^b y_n(\eta) h_0(\xi-\eta) d\eta \right|^2 d\xi , \qquad (3.57)$$

where F is the effective filter bandwidth in the frequency region

$$F = \frac{1}{v_F^2(0)} \int_{-\infty}^{+\infty} v_F^2(f) df \quad . \qquad (3.58)$$

We note that, in derivation of all relationships for the estimate $\phi_{nSF}(t)$, we assumed the white noise Y(t) and processes $Y_1(t)$, $Y_2(t)$, etc., reverting to zero outside the interval (a, b). It is evident that the same results are obtained in the case, when the random processes indicated decrease sufficiently quickly to infinity. If the processes do not decrease, realizations, abridged in interval (a, b), can be examined. In this case, the additional error of the estimate will be noted, only in a small vicinity of the ends of the interval, on the order of the effective bandwidth $\Delta T$ of weight function $w_L(t)$. /34

By use of the estimate of the second kind (3.29), an analogous theory can be developed for them. We are limited here, by the fact that we present the final form of the estimate of white noise intensity. With the use of the inverse Fourier transform of the convolution, from (3.30), we have

$$S_{\Delta T}(f, t_0) = \int_{-\infty}^{+\infty} \tilde{h}_{\Delta T}(t, t_0) x(t) e^{i2\pi f t} dt . \qquad (3.59)$$

By substitution of (3.59) in (3.29), we finally obtain

$$\varphi_{ssT}(t) = \int_{-\infty}^{+\infty} k_{sT}(f,0) \left| \int_{-\infty}^{+\infty} k_{\sigma T}(\eta, t)\, s(f-\eta)\, d\eta \right|^2 df =$$

$$= \int_{-\infty}^{+\infty} k_{sT}(f,0) \left| \int_{-\infty}^{+\infty} \tilde{k}_{\sigma T}(\xi, t)\, x(\xi)\, e^{i2\pi f \xi} d\xi \right|^2 df . \tag{3.60}$$

This estimate of the second kind has the same structure as the generalized estimate of the first kind (3.57), up to the transition to the time region. In turn, the estimate of the first kind (3.57) has the structure of the estimate of the second kind (3.60), up to the transition to the time region. In this manner, the estimates of the first and second kinds are dual, with respect to each other. It follows from this that all properties of estimates of the first kind can be reformulated for estimates of the second kind. From the fact that an estimate, for example, of the first kind for intensity has the same structure as an estimate of the second kind for spectral density, total symmetry of f and t follow, in obtaining estimates of the quadratic characteristics, by averaging over these variables. In other words, if random process X(t) is such that, in the time or frequency regions, it permits averaging over a large interval and, consequently, averaging can be carried out in a small interval over a different variable, for such a process the methods examined permit estimates of the quadratic characteristics to be obtained with small displacement and dispersion.

Now, let an actual, nonstationary, wideband process X(t), with correlation function $K_X(t_1, t_2)$, be given. In accordance with the preceding, for each white noise Y(t), with correlation function $K_Y(t_1, t_2) = b_Y(t_1)\delta(t_1 - t_2)$, there is an infinite set of sequences $Y_1(t)$, $Y_2(t)$, . . ., $Y_n(t)$, . . . of common random processes, which converge to it, in the sense of generalized functions. We examine in greater detail, which is important for the future, the question of the possibility of inclusion of a given broad band, nonstationary process in the sequence $Y_n(t)$ (n = 1, 2, . . .), as a term in it

$$X(t) = Y_m(t) ,$$

and also, how to estimate the closeness of X(t) to white noise Y(t).

As before, we will assume that all the processes considered differ from zero, only in a finite interval (a, b). We saw (3.35) and (1.38) that

$$\lim_{n \to \infty} \int_a^b \int_a^b K_{Y_n}(t_1, t_2) \psi_1(t_1) \psi_2(t_2) dt_1 dt_2 = \int_a^b b_Y(t_1) \psi_1(t_1) \psi_2(t_1) dt_1 , \tag{3.61}$$

where $\psi_1(t)$ and $\psi_2(t)$ are any functions of basic space $\mathcal{K}_1$. /35 Further, let $h(t, \xi)$ be a transition pulse function of a certain linear system. Subsequently, for simplification, we will assume that $h(t, \xi)$ is an infinitely differentiable function of both arguments, integrable over the entire plane $(t, \xi)$, at least with a square. (For physically realizable systems $h(t, \xi)$ identically reverts to zero in the first quadrant, and it undergoes a discontinuity at its boundary.) In many cases, the linear systems encountered in practice satisfy this requirement. With the assumptions made, for a fixed t, any $h(t, \xi)$ can be represented as the limit of a sequence of functions of $\mathcal{K}_1$, converging to $h(t, \xi)$, in any finite interval, uniformly, together with the derivatives of any order [2]. It follows from this that relationship (3.61) remains true for any two $h_1(t, \xi)$, $h_2(t, \xi)$, i.e.,

$$\lim_{n\to\infty}\int_a^b\int_a^b K_{Y_n}(\xi_1,\xi_2)\,h_1(t_1,\xi_1)\,h_2(t_2,\xi_2)\,d\xi_1 d\xi_2 = \int_a^b b_Y(\xi_1)\,h_1(t_1,\xi_1)\,h_2(t_2,\xi_1)\,d\xi_1 . \tag{3.62}$$

It is evident that, for any sequence $Y_n(\xi)$, this relationship (to the point, like (3.61)) cannot be written, for any finite n, in the form of an approximate equality. Actually, for any correlation function $K_{Y_n}(\xi_1, \xi_2)$, however small the effective correlation length, such a sharply changing $h_1(t, \xi)$ and $h_2(t, \xi)$ can be found, that the left side will differ as strongly as desired from the right. This means that the closeness of $K_{Y_n}(\xi_1, \xi_2)$ to $K_Y(\xi_1, \xi_2) = b_Y(\xi_1)\delta(\xi_1 - \xi_2)$, makes it sensible to consider the derived set of transition pulse functions only in a definite way.

We separate this set of conditions, so that each of the values in it

$$\| h_{12}(t_1, t_2 ; \xi_1, \xi_2)\|_m$$

is bounded by its constant, i.e.,

$$\| h_{12}(t_1, t_2 ; \xi_1, \xi_2)\|_m < C_m , \tag{3.63}$$

where

$$\| h_{12}(t_1,t_2;\xi_1,\xi_2)\|_m = max\{|h_{12}|,|h'_{12\xi}|,\dots|h^{(m)}_{12\xi}|\}, \tag{3.64}$$

$$h_{12}(t_1,t_2;\xi_1,\xi_2) = h_1(t_1,\xi_1)\,h_2(t_2,\xi_2)\,,$$
$$h'_{12\xi}=\frac{\partial^2 h_{12}}{\partial\xi_1\partial\xi_2}\,,\quad\dots,\quad h^{(m)}_{12\xi}=\frac{\partial^{2m} h_{12}}{(\partial\xi_1)^m(\partial\xi_2)^m}\,. \tag{3.65}$$

With the assumptions made, the limiting transition for the non-negative function

$$\Delta_{Y_n}(t_1,t_2) = \Big|\int_a^\ell\int_a^\ell K_{Y_n}(\xi_1,\xi_2)\,h_1(t_1,\xi_1)h_2(t_2,\xi_2)\,d\xi_1 d\xi_2 - \\ -\int_a^\ell b_Y(\xi_1)\,h_1(t_1,\xi_1)\,h_2(t_2,\xi_1)\,d\xi_1\Big| \tag{3.66}$$

can give the estimate [6]

$$\Delta_{Y_n}(t_1,t_2) \leqslant C_{Y_n}\| h_{12}(t_1,t_2;\xi_1,\xi_2)\|_1\,, \tag{3.67}$$

where the constant $C_{Y_n}$ is its own for each $Y_n$. Further, in an analogous manner, it can be shown that, in the case being considered, $C_{Y_n} \to 0$ uniformly in the separated set, i.e., for any $\varepsilon > 0$, such an N can be found that, for $n > N$,

$$\Delta_{Y_n}(t_1,t_2) \leqslant \varepsilon\, C_1 \tag{3.68}$$

where $C_1$ is the constant from relationship (3.63). In this manner, in the set of transition pulse functions separated, for a sufficiently large n, the limiting relationship (3.62) can be replaced by an approximate equality. /36

We now turn to the question of inclusion of a given non-stationary wideband process X(t) in the sequence $Y_n(t)$, converging to white noise. It should immediately be stated that, not concretely defining process X(t), it is practically impossible to answer the question and, what is more, to construct the specified sequence $Y_n(t)$, including process X(t) in it; therefore, we limit ourselves to qualitative considerations, based on the approximate asymptotic relationships, obtained for processes X(t), with a very short correlation length $\beta_0$. We will assume that

$$\beta_0 \ll T_x \quad \text{and} \quad \beta_0 < T_h\,, \tag{3.69}$$

where $T_x$ and $T_h$ are characteristic intervals of change in dispersion of equal $K_x(\xi, \xi)$ and of pulse function $h(t, \xi)$, of a certain set, fixed by relationships of the types (3.63) and (3.64). We introduce the designations

$$\alpha = (\xi_1 + \xi_2)/2 \quad , \quad \beta = \xi_2 - \xi_1 \quad ,$$
$$\overline{K}_x(\alpha, \beta) = K_x\left(\alpha + \frac{\beta}{2}, \alpha - \frac{\beta}{2}\right) \quad . \tag{3.70}$$

Then,

$$K_z(t_1, t_2) = \int_a^b\int_a^b K_x(\xi_1, \xi_2)\, h(t_1, \xi_1)\, h(t_2, \xi_2)\, d\xi_1 d\xi_2 =$$
$$= \int_0^{b-a} d\beta \int_{a+\beta/2}^{b-\beta/2} \overline{K}_x(\alpha, \beta)\, h\left(t_1, \alpha + \frac{\beta}{2}\right) h\left(t_2, \alpha - \frac{\beta}{2}\right) d\alpha + \int_{-(b-a)}^{c} d\beta \int_{a-\beta/2}^{b+\beta/2} \overline{K}_x(\alpha, \beta)\, h\left(t_1, \alpha + \frac{\beta}{2}\right) h\left(t_2, \alpha - \frac{\beta}{2}\right) d\alpha \approx \tag{3.71}$$
$$\approx \int_a^b h(t_1, \alpha)\, h(t_2, \alpha) \left[\int_{-(b-a)}^{b-a} \overline{K}_x(\alpha, \beta)\, d\beta\right] d\alpha =$$
$$= \int_a^b h(t_1, \alpha)\, h(t_2, \alpha)\, b_x(\alpha)\, d\alpha \quad ,$$

where

$$b_x(\alpha) = \int_{-(b-a)}^{b-a} \overline{K}_x(\alpha, \beta)\, d\beta \quad , \tag{3.72}$$

$K_z(t_1, t_2)$ is the correlation function of the response of a system with pulse function $h(t, \xi)$ to the action of process $X(\xi)$. It is evident that, for actual processes with a small correlation length $\beta_0$, function $b_x(\alpha) > 0$. In accordance with the preceding, this approximate equality will be true, for all pulse functions of the set formed. Of course, function $b_x(\xi)$, defined by relationship (3.72), is approximately interpreted as the effective intensity of the nonstationary wideband process considered $X(\xi)$ or, in accordance with (B.5), as have the instantaneous spectrum of this process. Thus, $K_x(\xi_1, \xi_2)$ can be represented approximately in the following form

$$K_x(\xi_1, \xi_2) = \overline{K}_x(\alpha, \beta) = b_x(\alpha)\, w_x(\alpha, \beta) \quad , \tag{3.73}$$

where

$$w_x(\alpha, \beta) = \frac{\overline{K}_x(\alpha,\beta)}{\int_{-(b-a)}^{b-a} \overline{K}_x(\alpha,\beta)\, d\beta} = \frac{\overline{K}_x(\alpha,\beta)}{b_x(\alpha)} \quad . \tag{3.74}$$

It is evident that, for any $\alpha$, /37

$$\int_{-\infty}^{+\infty} w_x(\alpha,\beta)\, d\beta = 1 \quad . \tag{3.75}$$

We introduce the sequence

$$w_{\mathcal{L}_n}(\alpha,\beta) = \mathcal{L}_n w_x(\alpha, \mathcal{L}_n\beta) \, , \tag{3.76}$$

where $\mathcal{L}_n \to \infty$ as $n \to \infty$. In accordance with the results of section 1, we see that, for any fixed $\alpha$, function $w_{\mathcal{L}_n}(\alpha, \beta)$ forms a $\delta$-form sequence, in which all functions are normalized, i.e., they satisfy the same condition (3.75) as $w_x(\alpha, \beta)$. By selecting $\mathcal{L}_n$ so that $\mathcal{L}_m = 0$, for $n = m$, we suppose

$$K_{Y_n}(\xi_1,\xi_2) = \overline{K}_{Y_n}(\alpha,\beta) = b_x(\alpha) w_{\mathcal{L}_n}(\alpha,\beta) = b_x(\alpha)\mathcal{L}_n w_x(\alpha,\mathcal{L}_n\beta) \, . \tag{3.77}$$

It is seen from relationship (3.76) that, as $n \to \infty$ ($\mathcal{L}_n \to \infty$), function $w_{\mathcal{L}_n}(\alpha, \beta)$ stops depending on $\alpha$, and we obtain, in the limit,

$$K_{Y_n}(\xi_1,\xi_2) = \overline{K}_{Y_n}(\alpha,\beta) \xrightarrow[n\to\infty]{} b_x(\alpha)\,\delta(\beta) \tag{3.78}$$

in the sense of the generalized functions. According to (3.73),

$$K_x(\xi_1,\xi_2) = \overline{K}_x(\alpha,\beta) = b_x(\alpha) w_x(\alpha,\beta) = K_{Y_m}(\xi_1,\xi_2) \, . \tag{3.79}$$

By the use of the approximations adopted, the following can be written

$$\int_a^b\int_a^b K_{Y_n}(\xi_1,\xi_2) f(\xi_1) f(\xi_2)\, d\xi_1 d\xi_2 \approx \int_a^b b_x(\alpha)[f(\alpha)]^2 d\alpha \geqslant 0 \quad . \tag{3.80}$$

In this manner, functions $K_{Y_n}(\xi_1, \xi_2)$ are at least asymptotically positively defined and, consequently, they can serve as correlation functions of certain random processes. It follows from this that, in accordance with (3.78), the necessary condition is fulfilled, for existence of sequence of random processes $Y_n(\xi)$ converging in the sense of generalized functions to a certain white noise $Y(\xi)$, in which $Y_m(\xi) = X(\xi)$.

We now consider how the closeness of process $X(t)$ to the limiting white noise $Y(t)$ can be estimated. In solution of practical problems, we usually are interested in the action of process $X(t)$ in linear systems, described by one or more families of transition pulse functions, which depend on a series of parameters. In the majority of cases, the carrier frequency $f_0$ and the effective bandwidth of the frequency characteristic of the system $\Delta f_0$ serve as these parameters. The transition pulse functions used $h_{f_0}(t, \xi; \Delta f_0)$, as a rule, are bounded, and growth of the modulus of their derivatives with increase in order is completely defined by parameters $f_0$ and $\Delta f_0$. Therefore, in order to include the family of transition pulse functions considered in a certain set (3.63), quantities $f_0$ and $\Delta f_0$ must be bounded. The characteristic interval $T_h$, in which the transition pulse function changes noticeably, can be estimated by the relationship

$$T_h \sim \min\{1/f_0,\ 1/\Delta f_0\}. \tag{3.81}$$

In deriving formula (3.71), we saw that, by fulfillment of the inequalities

$$\begin{aligned} &\rho_0 < T_h \quad \text{or} \quad \Delta f_{eff} > \max\{f_0, \Delta f_0\}, \\ &T_x \gg \rho_0 \quad \text{or} \quad T_x \gg 1/\Delta f_{eff}, \end{aligned} \tag{3.82}$$

where $\Delta f_{eff}$ is the effective width of the amplitude spectrum of the process, and nonstationary wideband process $X(t)$ is close to white noise, with intensity $b_x(t)$ (see 3.72). Closeness is understood in the sense that its action on any linear system with a transition function belonging to the set formed, with sufficient accuracy, determined by formula (3.68), is similar to the action of the equivalent white noise. In this manner, these inequalities (3.82) give the qualitative conditions of closeness of wideband process $X(t)$ to the limiting white noise and, consequently, the truth of the approximate expression for the correlation function of the response of system (3.71a) /38

$$K_z(t_1, t_2) \approx \int_{\alpha}^{t} h(t_1, \lambda)\, h(t_2, \lambda)\, b_x(\lambda)\, d\lambda$$

with a sufficient degree of accuracy. A quantitative refinement of these inequalities is given by means of study of concrete model processes in the subsequent sections.

We note in conclusion that the condition of normalization of all functions $w_{\delta_n}(\alpha, \beta)$ (see 3.76), in representation of the correlation functions of wideband process (3.73) is very significant. In fact, we can write

$$K_x(\xi_1, \xi_2) = \hat{b}_x(\alpha)\, \hat{w}_{\delta_m}(\alpha, \beta) \tag{3.83}$$

by many methods, with the use of different $\delta$-form sequences $\hat{w}_{\delta m}(\alpha, \beta)$. As $n \to \infty$, each such sequence will correspond to its sequence of random processes $\hat{Y}_n(\xi)$, converging to white noise $\hat{Y}(\xi)$, with intensity $\hat{b}_x(\xi)$. If the sequence is not normalized, i.e.,

$$\int_{-\infty}^{+\infty} \hat{w}_{\delta_n}(\alpha, \beta)\, d\beta = q_n\, , \qquad \lim_{n \to \infty} q_n = 1\, , \tag{3.84}$$

then, by carrying out normalization, from (3.83), we obtain

$$K_x(\xi_1, \xi_2) = b_{mx}(\alpha)\, w_{\delta_m}(\alpha, \beta)\, ,$$
$$b_{nx}(\alpha) = \hat{b}_x(\alpha)\, q_n\, , \quad w_{\delta_n}(\alpha, \beta) = \hat{w}_{\delta_n}(\alpha, \beta)/q_n\, ,$$
$$\lim_{n \to \infty} b_{nx}(\alpha) = \hat{b}_x(\alpha)\, .$$

By substitution of this expression for $K_x(\xi_1, \xi_2)$ in (3.71a), we obtain

$$K_z(t_1, t_2) \approx q_m \int_a^b \hat{b}_x(\alpha)\, h(t_1, \alpha)\, h(t_2, \alpha)\, d\alpha\, ,$$

where $q_m$, in principle, can be any number. Thus, for the function $\hat{b}_x(\alpha)$ itself, relationship (3.71a) is not fulfilled, and this means that wideband process $X(\xi)$ is far from the limiting white noise $\hat{Y}(\xi)$ with intensity $\hat{b}_x(\alpha)$. It is evident that the process $X(\xi)$ must be included in a sequence, in which the limiting white noise is close to wideband process $X(\xi)$. Fortunately, the theory of estimation of white noise intensity stated above shows that, if we analyze the realization of a certain wideband process by means of the methods developed, we always automatically obtain (good or poor, depending on conditions) an intensity of precisely the white noise closest to our process.

## CHAPTER II

## APPLICATION OF ESTIMATES TO ANALYSIS OF NONSTATIONARY WIDEBAND PROCESSES

### 4. Modeling of Nonstationary Random Processes

We consider a family of nonstationary random processes, which /39 depend on a series of parameters, and we show that, in definition of the relationships between these parameters, processes can be obtained, as close as desired (in the sense of section 3) to nonstationary white noise. Moreover, we introduce the necessary formulas for calculation of individual realizations of all possible wideband noises, which are nonstationary and wideband to different degrees.

As a model random process, we take an expression of the type

$$X(t) = R(t)H(t)\cos(2\pi f_n t + G(t)), \tag{4.1}$$

where $t > -\infty$ is the time, $f_n \geq 0$ is a fixed carrier frequency, $R(t)$ is an arbitrary, dimensionless, determinate function and $H(t)$ and $G(t)$ are random processes (amplitude and phase), with realizations $h(t)$ and $g(t)$, respectively. Amplitude $H(t)$ has the dimensionality of process [X], and phase $G(t)$ is a dimensionless function. Realization $x(t)$ of process $X(t)$ is written in the form

$$x(t) = R(t)\,h(t)\cos(2\pi f_n t + g(t)). \tag{4.2}$$

In order to have the possibility of constructing realization $x(t)$ from point to point, with $\Delta t$ intervals, we take known, stationary, independent Markoff processes as $H(t)$ and $G(t)$ [16]. The conditional probability of transition of the amplitude fluctuation

$$v_h(h(t+\tau)\,|\,\tau, h(t)) = \frac{1}{\sigma_h(\tau)\sqrt{2\pi}} \exp\left\{-\frac{[h(t+\tau) - m(t,\tau)]^2}{2\sigma_h^2(\tau)}\right\}, \tag{4.3}$$

$$-\infty < h(t+\tau) < +\infty,$$

where

$$m(t,\tau) = h(t)\exp(-p\tau)\,, \tag{4.4}$$

$$\sigma_h^2(\tau) = r^2(1 - \exp(-2p\tau))\,, \tag{4.5}$$

r > 0 and p > 0 are fixed parameters. The conditional probability of the phase transition

$$v_g(g(t+\tau)|\tau, g(t)) = \begin{cases} \frac{1}{2\pi}\left\{1 + 2\sum_{l=1}^{\infty} e^{-l^2 q\tau}\cos[l(g(t+\tau) - g(t))]\right\} \\ \qquad\qquad \text{for } 0 \le g(t+\tau) \le 2\pi\,, \\ 0 \quad \text{for } g(t+\tau) < 0 \;\blacksquare\; g(t+\tau) > 2\pi\,, \end{cases}$$

where q > 0 is a fixed parameter.

As $\tau \to \infty$, the conditional probability densities are converted into stationary, unidimensional distributions

$$w_{1h}(h(t)) = \frac{1}{r\sqrt{2\pi}}\exp\left(-\frac{h^2(t)}{2r^2}\right), \quad -\infty < h(t) < +\infty\,, \tag{4.7}$$

$$w_{1g}(g(t)) = \begin{cases} \frac{1}{2\pi} \text{ for } 0 \le g(t) \le 2\pi\,, \\ 0 \text{ for } g(t) < 0\,,\; g(t) > 2\pi\;; \end{cases} \tag{4.8}$$

since, for a Markoff process, knowing the probability of tran- /40
sition and the unidimensional probability, any n-dimensional distribution density can be written, it is easy to show that all n-dimensional distributions are uniform over time. In this manner, processes H(t) and G(t) are stationary in the narrow sense.

We note in passing that, if function R(t) = R = const, since processes H(t) and G(t) are independent by definition, and G(t) is distributed uniformly in the interval [0.2 π], arkoff process X(t) will be stationary in the narrow sense. The mathematical expectation, correlation function and spectral density, in this case, have the form:

$$m_x = 0\,, \tag{4.9}$$

$$K_x(\tau) = R^2\frac{r^2}{2}e^{-(p+q)|\tau|}\cos 2\pi f_H\tau\,, \tag{4.10}$$

$$W_x(f) = R^2 r^2 (p+q)\left\{\frac{1}{4\pi^2(f-f_n)^2+(p+q)^2} + \frac{1}{4\pi^2(f+f_n)^2+(p+q)^2}\right\}. \qquad (4.11)$$

To avoid misunderstanding, we note that the definition of spectral density of stationary processes given in Chapter I (2.5), which is true for both positive and negative frequencies, differs (is twice as small) from that which we use in practice, and which is true only for positive frequencies. However, we will not change the designation $W_x(f)$ for spectral density. Just as was done in [17], the ergodicity relative to the mathematical expectation and the correlation moment of stationary process X(t) can be shown.

In the general case, there is no difficulty in calculating the mathematical expectation, correlation function and instantaneous power spectrum:

$$m_x = 0, \qquad (4.12)$$

$$K_x\left(t-\frac{\tau}{2}, t+\frac{\tau}{2}\right) = R\left(t-\frac{\tau}{2}\right)R\left(t+\frac{\tau}{2}\right)\frac{r^2}{2}e^{-(p+q)|\tau|}\cos 2\pi f_n \tau, \qquad (4.13)$$

$$W_x(f,t) = 4R(t)\int_0^\infty R(t-\tau)\frac{r^2}{2}e^{-(p+q)\tau}\cos(2\pi f_n\tau)\cos(2\pi f\tau)\,d\tau. \qquad (4.14)$$

We now consider conditions under which processes (4.1-4.8) are close to white noise, in the broad sense. We assume in formula (4.13)

$$r^2 = p + q = \mathfrak{D}. \qquad (4.15)$$

The quantities $r^2$, p and q have different physical meanings: $r^2$ is the dimension of power $[X^2]$ and p and q are frequency dimensions [f]; therefore, equality (4.15) should be understood as

$$r^2[X^2] = k(p+q)[f], \qquad (4.16)$$

where k is a proportionality factor, having the dimension of the spectral power density

$$[k] = [X^2/f]. \qquad (4.17)$$

subsequently, we will consider that $\mathfrak{D}$ has the dimensions of frequency, and we will use the equalities /41

$$\eta^2 = \kappa \mathcal{D}\ , \quad p = \mathcal{D} - q\ , \tag{4.18}$$

where the value of coefficient k will depend on the concrete physical scales of values of functions X(t) and frequency f.

In accordance with the result . ted at the end of section 3, in order to find the white noise intensity, to which the model process converges, as $\mathcal{D} \to \infty$, the correlation function must be represented, in the form of the product of a simple function and the normalized element of the $\delta$-form sequence. By calculating the integral

$$\frac{\kappa \mathcal{D}}{2} \int_{-\infty}^{+\infty} e^{-\mathcal{D}|\tau|} \cos 2\pi f_M \tau \, d\tau = \frac{\kappa \mathcal{D}^2}{\mathcal{D}^2 + 4\pi^2 f_M^2}\ ,$$

in conformance with (4.13) and (4.18), we obtain

$$K_x\left(t - \frac{\tau}{2}, t + \frac{\tau}{2}\right) = R\left(t - \frac{\tau}{2}\right) R\left(t + \frac{\tau}{2}\right) \frac{\kappa \mathcal{D}^2}{\mathcal{D}^2 + 4\pi^2 f_M^2} w_{\mathcal{D}}(\tau)\ , \tag{4.19}$$

where

$$w_{\mathcal{D}}(\tau) = \frac{\mathcal{D}^2 + 4\pi^2 f_M^2}{\kappa \mathcal{D}^2} \frac{\mathcal{D}}{2} e^{-\mathcal{D}|\tau|} \cos 2\pi f_M \tau$$

is the normalized element of the $\delta$-form sequence, i.e.,

$$w_{\mathcal{D}} \to \delta\ , \quad \int_{-\infty}^{+\infty} w_{\mathcal{D}}(\tau)\, d\tau = 1\ .$$

Since $\delta$-form function $w_{\mathcal{D}}(\tau)$ differs from zero only at small $|\tau|$

$$K_x\left(t - \frac{\tau}{2}, t + \frac{\tau}{2}\right) = \frac{\kappa \mathcal{D} R^2(t)}{\mathcal{D}^2 + 4\pi^2 f_M^2} w_{\mathcal{D}}(\tau) = b(t) w_{\mathcal{D}}(\tau)\ , \tag{4.20}$$

$$W_x(f, t) = 2\, b(t) = \frac{2\kappa \mathcal{D} R^2(t)}{\mathcal{D}^2 + 4\pi^2 f_M^2} \tag{4.21}$$

and, consequently, function $b(t) = \kappa \mathcal{D} R^2(t)/(\mathcal{D}^2 + 4\pi^2 f_M^2)$ is the intensity of the nonstationary white noise, which is obtained from the initial model process (4.1-4.8), by fulfillment of condition (4.18) and as $\mathcal{D} \to \infty$.

We attempt to determine what the condition $\mathcal{D} \to \infty$ means in practice. Initially, we examine the case, when $R^2(t) = R^2 = \text{const}$, i.e., the process is stationary. By turning to the formula for spectral density (4.11), with (4.18) taken into account,

$$W_x(f) = \kappa R^2 \left\{ \frac{\mathcal{D}^2}{4\pi^2 (f - f_H)^2 + \mathcal{D}^2} + \frac{\mathcal{D}^2}{4\pi^2 (f + f_H)^2 + \mathcal{D}^2} \right\} \tag{4.22}$$

it is easy to show that the value of $\mathcal{D}$ is proportional to the effective bandwidth $\Delta f_{eff}$ of process X(t). Actually, for a spectral density which is symmetrical relative $f_H$, we have

$$\Delta f_{eff} = \frac{\sigma_x^2}{W_{max}} \tag{4.23}$$

where

$$\sigma_x^2 = \kappa R^2 \mathcal{D} / 2 \tag{4.24}$$

is the dispersion of the process, and /42

$$W_{max} = W_x(f_H) \approx \kappa R^2 \tag{4.25}$$

is the maximum value of the spectral density. From this, $\Delta f_{eff} \approx \mathcal{D}/2$ . The spectral densities of wideband processes have a strongly nonsymmetrical shape relative to $f_H$; therefore, the effective band for them is approximately half as large

$$\Delta f_{eff} \approx \frac{\mathcal{D}}{4} \tag{4.26}$$

and, consequently,

$$W_x(f) = W_{max} \approx \frac{2\kappa \mathcal{D} R^2}{\mathcal{D}^2 + 4\pi^2 f_H^2} \quad . \tag{4.27}$$

It is known that wideband processes are defined by the condition $\Delta f_{eff} \geq f_H$, or, in our case,

$$\mathcal{D} \geqslant 4 f_H \tag{4.28}$$

In particular, for $f_H = 0$, there will be a large band process X(t), at any $\mathcal{D} > 0$.

As should have been expected, it is seen from formulas (4.24) and (4.26) that $\mathcal{D} \to \infty$ entails an increase in dispersion and broadening the effective band of the process, with unchanged intensity $b = \kappa \mathcal{D} R^2/(\mathcal{D}^2 + 4\pi^2 f_H^2)$.

It is evident that, under actual conditions, the degree of closeness of wideband process X(t) to white noise, in the frequency region from 0 to $\Delta f_{eff}$, depends on the relationship between the duration of observation of the process (the characteristic time scale of the problem) $T_x$ and the time correlation $\beta_0$. This relationship can be expressed in the form

$$\beta_0 \ll T_x \quad . \tag{4.29}$$

By somehow defining the time correlation $\beta_0$, for example, as

$$\frac{K_x(\tau)}{K_x(0)} = e^{-\mathcal{D}|\tau|} \cos 2\pi f_H \tau \leq \varepsilon \tag{4.30}$$

for all $\tau \geq \beta_0$, where $\varepsilon > 0$ is some small number ($\varepsilon \ll 1$), a dependence can be obtained between parameters $\mathcal{D}$ and the time correlation $\beta_0$

$$\beta_0 = \frac{1}{\mathcal{D}} \ln\left(\frac{1}{\varepsilon}\right) . \tag{4.31}$$

As $\mathcal{D} \to \infty$, $\beta_0 \to 0$. Condition (4.29) takes the form

$$\frac{1}{\mathcal{D}} \ln\left(\frac{1}{\varepsilon}\right) \ll T_x \tag{4.32a}$$

or, with (4.26) taken into account,

$$\frac{1}{4\Delta f_{eff}} \ln\left(\frac{1}{\varepsilon}\right) \ll T_x \quad . \tag{4.32b}$$

Finally, (4.18), (4.28) and (4.32) give us the relationship for selection of the necessary models from the entire family of processes (4.1-4.8).

In the general case of nonstationary noise, relationships (4.18) and (4.28) are preserved, and quantity $T_x$ in formula (4.32) takes on the meaning of the characteristic scale of the nonstationary behavior of intensity b(t).

In section 6, we report further explanation of the practical content of condition (4.32), as well as study of certain other properties of model and actual noises, and here, we /43 proceed to construction of individual realizations of model processes by means of digital computer.

## 5. Obtaining Realizations of Model Processes by the Monte Carlo Method, by Means of Digital Computer

To obtain individual realizations of random process (4.1-4.8) in time interval $0 \le t \le T$, we use the normal procedure of the Monte Carlo method [3]. Let $\Delta t$ be the digitization step of process $X(t)$ and $n$ be the number of points of realization of the process in interval $T = n\Delta t$. Realizations of the amplitude $H(t)$ and phase $G(t)$ fluctuations and process $X(t)$ at discrete moments of time $0, \Delta t, 2\Delta t, \ldots, (n-1)\Delta t$, are designated, respectively, $h_j$, $g_j$, $x_j$ ($j = 1, 2, \ldots n$).

The unidimensional density distributions, with (4.18) taken into account, are written in the form:

$$w_{1h}(h_j) = \frac{1}{\sqrt{\kappa D\, 2\pi}} \exp\left(-\frac{h_j^2}{2\kappa D}\right), \quad -\infty < h_j < +\infty, \tag{5.1}$$

$$w_{1g}(g_j) = \begin{cases} \frac{1}{2\pi} & \text{for } 0 \le g_j \le 2\pi, \\ 0 & \text{for } g_j < 0,\ g_j > 2\pi, \end{cases} \tag{5.2}$$

and the conditional probabilities of the transition will be:

$$v_h(h_{j+1} \mid \Delta t, h_j) = \frac{1}{\sigma_h \sqrt{2\pi}} \exp\left\{-\frac{(h_{j+1} - m_j)^2}{2\sigma_h^2}\right\}, \quad -\infty < h_{j+1} < +\infty, \tag{5.3}$$

where

$$m_j = h_j \exp(-(D - q)\Delta t), \tag{5.4}$$

$$\sigma_h^2 = \kappa D\left(1 - \exp(-2(D - q)\Delta t)\right), \tag{5.5}$$

$$v_g(g_{j+1} \mid \Delta t, g_j) = \begin{cases} \frac{1}{2\pi}\left\{1 + 2\sum_{l=1}^{\infty} e^{-l^2 q \Delta t} \cos[l(g_{j+1} - g_j)]\right\} & \text{for } 0 \le g_{j+1} \le 2\pi, \\ 0 & \text{for } g_{j+1} < 0,\ g_{j+1} > 2\pi. \end{cases} \tag{5.6}$$

The amplitude fluctuations have normal distribution (5.1) and (5.3); therefore, calculation of sequence $h_j$ can be carried out as

$$h_1 = \nu_1 \sqrt{D}\ , \quad h_j = \nu_j \sigma_k + h_{j-1} m_j\ , \quad j = 2, 3, \ldots, n, \tag{5.7}$$

where $\nu_j$ is a sequence of independent pseudorandom numbers, obtained by programming, which are normally distributed over interval [0, 1].

For calculation of sequence $g_j$ ~~for~~ $j = 2, 3, \ldots, n$, the converse of the known theorem [3], on transformation of random quantity G, with assigned distribution pattern w(g), to random quantity P, uniformly distributed over interval [0, 1], can be used

$$P = \int_{-\infty}^{G} w(g)\, dg\ . \tag{5.8}$$

The standard procedure for plotting numbers $g_j$, with a fixed distribution pattern $v_g(g_{j+1}|\Delta t,\ g_j)$ from a sequence of pseudorandom numbers $P_j$ (obtained just like $\nu_j$, by programming), uniformly distributed over interval [0, 1], consists of solution for $g_j$, of equation /44

$$\rho_j = \frac{1}{2\pi} \int_0^{g_j} \{1 + 2 \sum_{l=1}^{\infty} e^{-l^2 q \Delta t} \cos[l(g - g_{j-1})]\}\, dg\ , \quad j = 2, 3, \ldots, n. \tag{5.9}$$

By taking the integral, we obtain the following transcendental equation, which can be solved digitally for each $g_j$, for example, by the secant method

$$\rho_j = \frac{1}{2\pi} g_j + \frac{1}{\pi} \sum_{l=1}^{\infty} e^{-l^2 q \Delta t} \left\{ \frac{\sin l(g_j - g_{j-1})}{l} + \frac{\sin l g_{j-1}}{l} \right\}, \quad j = 2, 3, \ldots, n. \tag{5.10}$$

By adding to the sequence obtained $g_2, g_3, \ldots, g_n$, the value

$$g_1 = 2\pi \rho_1\ , \tag{5.11}$$

we obtain the following final expression, for realization of x(t) of random process X(t), at discrete moments of time

$$x_j = R_j h_j \cos[2\pi f_m \Delta t(j-1) + g_j] , \qquad j = 1, 2, 3, \ldots, n , \tag{5.12}$$

where $R_j$ are values of function $R(t)$ at moments of time 0, $\Delta t$, $2\Delta t$, . . . , $(n - 1)\Delta t$.

To obtain different independent realizations of the same process, it is sufficient to change number $\ell_0$ of the initial pseudorandom number $P_1$ and number $m_0$ of the initial pseudorandom number $\nu_1$ simultaneously, in the digital computer generating programs, by a quantity larger than n:

$$\ell_1 > \ell_0 + n \quad , \quad m_1 > m_0 + n . \tag{5.13}$$

We assume that programs which generate normally distributed numbers $\nu_j$ and uniformly distributed numbers are independent. In particular, while in the program generating normally distributed numbers, the sum k of sequential numbers $P_j$ are used, first, number $m_0$ should be selected from the condition

$$m_0 > (\ell_0 + n)/k \tag{5.14}$$

and, second, to obtain another (independent) realization of numbers $\ell_1$ and $m_1$, the following conditions must be satisfied

$$\ell_1 > (m_0 + n)k \quad , \quad m_1 > (\ell_1 + n)/k . \tag{5.15}$$

A program for calculation of normally distributed pseudorandom numbers, in which number $k = 20$, was used in this work.

A few words on selection of the digitization step $\Delta t$ and parameter q. The digitization step should be selected from the condition

$$\Delta t \ll \frac{1}{2\Delta f_{eff}} \quad , \tag{5.16}$$

since, in changing from the continuous process to a random sequence, superposition of the spectral densities on each frequency f takes place, at points $(k/\Delta t + f)$ and $(k/\Delta t - f)$ ($k = 1, 2,$ /45
. . . , $\infty$). For a wideband, random process, with discrete time value $1/(2\Delta t)$, the highest ("infinitely" high) frequency has meaning, but, as was shown in section 4, quantity $\mathcal{D}$ has the same meaning; therefore, it is sufficient to take

$$\Delta t \approx \frac{1}{2\mathcal{D}} \cdot \tag{5.17}$$

The parameter should be selected, so that the value of $2(\mathcal{D} - q)\Delta t$ is on the order of unity

$$2(\mathcal{D} - q)\Delta t \approx 1, \tag{5.18}$$

since, otherwise, dispersion $\sigma_h^2$ (see 5.5) of the amplitude fluctuation turns out to be very small, i.e., the total level of the realization will, for a long period of time t, depend on the initial value of the amplitude $h_1$ (see 5.7). Consequently, process X(t) will have poor ergodicity with respect to the correlation moment. An example, in which $2(\mathcal{D} - q)\Delta t = 0.04$, is presented in Fig. 1. The estimate of the spectral density, calculated with realization length T = 100 sec and averaging interval $\Delta f = 0.1$ Hz, which usually gives good results (see [17]), is strongly understated. To obtain a correct result, a realization of significantly greater duration must be used. We also note that it is advisable that not too small (on the order of 0.1) a value of $q\Delta t$ be taken, since, otherwise, the series over $\ell$ in formula (5.10) will converge slowly. It is clear that this difficulty is not fundamental, but computational.

## 6. Practical Questions of Similar Transformation of Wideband Processes

In the solution of practical problems, connected with analysis of the frequency-time structure of random processes and their actions on dynamic systems, the numerical values of the following basic characteristics will be known ahead of time, as a rule: effective bandwidth $\Delta f_{eff}$, characteristic scale of transiency $T_x$ and the actual limits for the region of permissible values of process X(t), for example, for normal centering processes, the value of $3\sigma_x$.

As was noted in section 4, the basic qualitative condition, under which a wideband process ($\Delta f_{eff} > f_H$) in interval $T_x$ can be approximately considered to be white noise, in the frequency range from f = 0 to $f \approx \Delta f_{eff}$, has the form

$$\beta_0 \ll T_x \tag{6.1}$$

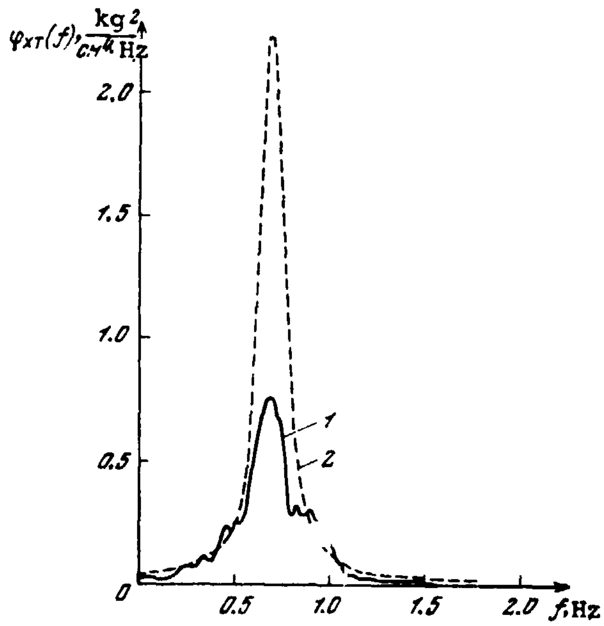


Fig. 1. Estimate of spectral density, calculated from realization of model narrow band process, having "weak" ergodicity relative to correlation moment ($\Delta t$=0.1 sec, R=1.0, $r^2$=1.0 $kg^2/cm^4$, p=0.2 Hz, q=0.25 Hz, f=0.7 Hz, T=100 sec, $\Delta f$=0.1 Hz, $\ell_0$=20,000, $m_0$=1); 1. estimate $\phi_{xT}(f)$; 2. theoretical spectral density $W_x(f)$.

The time correlation $\beta_0$ for wideband processes is inversely proportional to the bandwidth $\Delta f_{eff}$ /46

$$\beta_0 = A \frac{1}{\Delta f_{eff}}, \qquad (6.2)$$

A is a proportionality factor, which, for model processes based on (4.26) and (4.31), is given by the formula

$$A = \frac{1}{4} \ln\left(\frac{1}{\varepsilon}\right) \qquad (6.3)$$

and it is on the order of unity. It is clear that formula (6.3) can be used for the majority of actual processes.

Condition (6.1) takes the form

$$T_x \Delta f_{eff} \gg 1 \,. \qquad (6.4)$$

Subsequently, we will explain the practical content of this relationship by examples.

Now, we give attention to the fact that, under this condition, no value scale of function X(t) is included. Therefore, we will call the two random processes similar in the spectral sense, if they have identical dimensionless coefficients

$$P_1 = T_x \Delta f_{eff} \;, \; P_2 = f_H / \Delta f_{eff} \qquad (6.5)$$

regardless of the scale of values of these processes.

The model random processes considered in section 4 can well approximate actual processes and, consequently, they can be used for development of methods of selection of the optimum parameters necessary for analysis of experimental data. In order to completely define subsequent reasoning, we will consider that the time is measured in seconds (sec), the frequency in Hertz (Hz), and that the random processes are oscillations of gas pressure, measured in ($kg/cm^2$).

We present the example of a model wideband noise, with specification of all the quantitative relationships between its parameters, and we consider the question of transformations which are similar in the spectral sense, of the parameters this noise. For simplicity, we use a stationary noise ($R(t) = R = \text{const}$). Let it be known that the effective bandwidth of the random noise is

$$\Delta f_{eff} \sim 1.38 \text{ Hz} - 1.90 \text{ Hz}, \tag{6.6}$$

and the minimum characteristic time scale of the problem is reckoned in tens of seconds

$$T_x \sim 20 \text{ sec} - 50 \text{ sec}. \tag{6.7}$$

We also assume that the approximate value of the mean square level of the oscillations

$$\sigma_x \sim 0.7 \text{ kg/cm}^2 - 0.8 \text{ kg/cm}^2 \tag{6.8}$$

and the mean square value of the overshoot distribution

$$\sigma_\ell \sim 1.85 \text{ kg/cm}^2 - 2.00 \text{ kg/cm}^2. \tag{6.9}$$

are known. It is required that a noise model be selected, of type (4.1-4.8), and that the extent to which it will be close to white noise in the interval $T_x$ and in the frequency rates from $f = 0$ Hz to $f = \Delta f_{eff}$Hz be determined.

First of all, by the use of (4.26), we obtain

$$\mathcal{D} = 5.5 \text{ Hz} \tag{6.10}$$

In order that, in the interval $[0, \Delta f_{eff}]$, the spectral noise level be close to a constant value, in accordance with (4.28), we use

$$f_H = 0.7 \text{ Hz}. \tag{6.11}$$

The digitization step $\Delta t$, on the basis of (5.17), will be /47

$$\Delta t = 0.1 \text{ sec} \tag{6.12}$$

Further, from (4.24), we obtain the value of the frequency average intensity

$$\ell = \kappa R^2 = \frac{2\sigma_x^2}{\mathfrak{D}} = 0.091 \frac{\text{kg}^2}{\text{cm}^4\,\text{Hz}} \tag{6.13}$$

Assuming $2\cdot(\mathfrak{D} - q)\cdot\Delta t = 1$ (see 5.18), we determine

$$q = 0.45 \text{ Hz}, \tag{6.14}$$

and, from (5.5), it is easy to obtain

$$\kappa = \frac{\sigma_k^2}{\mathfrak{D}(1 - \exp(-2(\mathfrak{D}-q)\Delta t))} = 1 \frac{\text{kg}^2}{\text{cm}^4\,\text{Hz}} \tag{6.15}$$

It follows from (6.13) and (6.15), that

$$R = 0.426. \tag{6.16}$$

We calculate realization (5.12) of a wideband, discrete, random process, with parameters (6.10-6.16), in time interval $T = 100$ sec. A fragment of realization of this process ($\ell_0 = 20{,}000$, $m_0 = 1$) is presented in Fig. 2 and, in Figs. 3 and 4, estimates of the spectral density and intensity. The theoretical spectral densities for a continuous random process and for a random sequence are plotted in Fig. 3 and, in Fig. 4, the theoretical intensity. From comparison of the curves in Fig. 3, it is seen that, in the frequency region of interest to us, from $f = 0$ Hz to $f = \Delta f_{eff} \approx 1.38$ Hz, the effect of the superimposition is small, and the estimate, calculated for the random sequence, over the entire frequency range [0 Hz, 5 Hz] well describes the corresponding theoretical curve. The estimate of the intensity also is close to the value sought (see Fig. 4). Based on this, it can be concluded that this wideband model process is close to white noise, in the interval $T_x \sim 20$ sec - 50 sec, and in the frequency range from 0 Hz to 1.38-1.90 Hz. /48

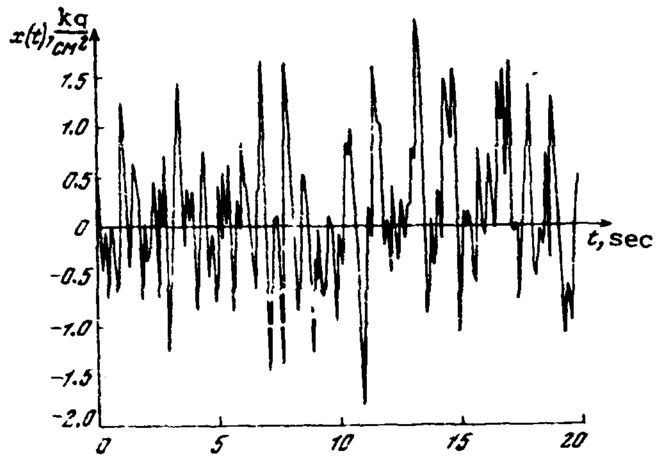


Fig. 2. Realization of model stationary noise with random amplitude and phase ($\Delta t$=0.1 sec, R=0.426, $\mathfrak{D}$=5.5 Hz, q=0.45 Hz, $f_H$=0.7 Hz, k=1 kg$^2$/(cm$^4$·Hz), $\ell_0$=20,000, $m_0$=1).

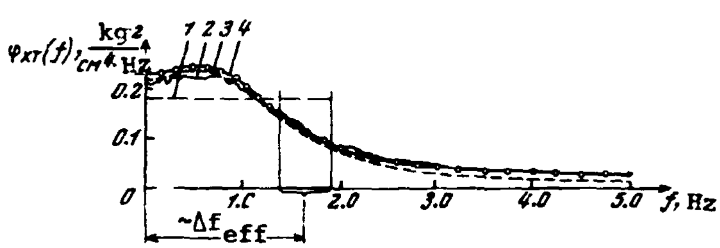


Fig. 3. Estimate of spectral density of stationary noise ($\Delta t$=0.1 sec, R = 0.426, $\mathfrak{D}$ =5.5 Hz, q=0.45 Hz, $f_H$=0.7 Hz, k=1 kg$^2$/(cm$^4$·Hz), T=100 sec, $\Delta f$=1.0 Hz, $\ell_0$=20,000, $m_0$=1); 1. double theoretical intensity, 2 b=2 kR$^2$; 2. estimate $\phi_{XT}(f)$; 3. theoretical spectral density of continuous process $W_x(f)$; 4. theoretical spectral density of discrete process $W_{xn}(f)$.

Assuming $T_x$ = 20 sec, we calculate the coefficients of similarity $P_1$ and $P_2$ (6.5)

$$P_1 \approx 28, \quad P_2 \approx 0.51 \qquad (6.17)$$

and we consider two processes, similar to this in the spectral sense, the effective frequency bands of which are equal to

$$\Delta f_{eff} \sim 55 \text{ Hz} - 76 \text{ Hz}, \qquad (6.18)$$

$$\Delta f_{eff} \sim 1.38 \cdot 10^6 \text{ Hz} - 19.0\ 10^6 \text{ Hz}, \qquad (6.19)$$

and the values of $\sigma_x$ and $\sigma_h$, as before, are

$$\sigma_x \sim 0.7 \text{ kg/cm}^2 - 0.8 \text{ kg/cm}^2, \qquad (6.20)$$

$$\sigma_h \sim 1.8 \text{ kg/cm}^2 - 2.00 \text{ kg/cm}^2. \qquad (6.21)$$

Retaining coefficients of similarity $P_1$ and $P_2$, for the first process, we obtain:

$$\begin{gathered} T_x \sim 0.51 \text{ sec} - 3.7 \text{ sec}, \\ f_H \sim 27.5 \text{ Hz} - 38 \text{ Hz}, \\ \mathfrak{D} = 220 \text{ Hz}, \quad \Delta t = 0.0025 \text{ sec}, \\ b = 0.227\ 10^{-2} \text{ kg/(cm}^4\text{ Hz)}, \quad q = 20 \text{ Hz}, \\ k = 0.025 \text{ kg}^2\text{/(cm}^4\text{ Hz)}, \quad R = 0.426 \end{gathered} \qquad (6.22)$$

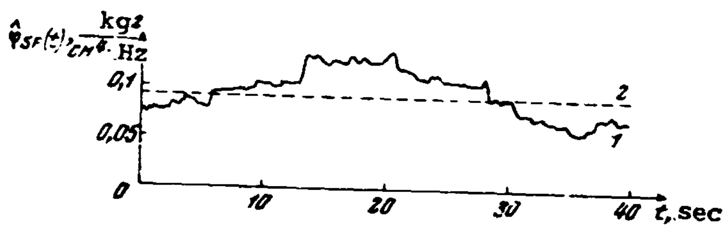


Fig. 4. Estimate of stationary noise intensity ($\delta T$=0.3 sec, F≈3.33 Hz, $\Delta T$=15 sec): 1. estimate $\hat{\phi}_{SF}(t)$; 2. theoretical intensity b.

and, for the second process:

$$T_x \sim 2\cdot 10^{-6}\,\text{sec} - 5\cdot 10^{-6}\,\text{sec},\quad f_H \sim 7\cdot 10^{6}\,\text{Hz} - 9.5\cdot 10^{6}\,\text{Hz},\quad \mathcal{D} = 55\cdot 10^{6}\,\text{Hz},\ \Delta t = 10^{-8}\,\text{sec},\quad b = 0.091\cdot 10^{-7}\,\text{kg}^2/(\text{cm}^4\,\text{Hz}),\quad q = 4.5\cdot 10^{6}\,\text{Hz},\ K = 10^{-7}\,\text{kg}^2/(\text{cm}^4\,\text{Hz}),\quad R = 0.426. \tag{6.23}$$

We present an example, which shows that, with use of the scale of the region of values of random process X(t), the form of the spectrum remains unchanged (coefficients of similarity $P_1$ and $P_2$ are retained), and only the intensity scale changes. /49 For the second process (6.19-6.21, 6.23), let $\sigma_x$ be increased to a value of

$$\sigma_x \sim 0.52\cdot 10^4\,\text{kg/cm}^2 - 0.59\cdot 10^4\,\text{kg/cm}^2 \tag{6.24}$$

and, correspondingly,

$$\sigma_k \sim 1.38\cdot 10^4\,\text{kg/cm}^2 - 1.47\cdot 10^4\,\text{kg/cm}^2 \tag{6.25}$$

in which, as before:

$$\Delta f_{eff} \sim 13.8\cdot 10^6\,\text{Hz} - 19.0\cdot 10^6\,\text{Hz},\quad T \sim 2\cdot 10^6\,\text{sec} - 5\cdot 10^6\,\text{sec},\quad f_H \sim 7\cdot 10^6\,\text{Hz} - 9.5\cdot 10^6\,\text{Hz}. \tag{6.26}$$

From formulas (4.26), (5.17), (4.24), (5.18) and (5.5), we obtain

$$\mathcal{D} = 55\cdot 10^6\,\text{Hz},\quad \Delta t = 10^{-8}\,\text{sec},\quad b = 0{,}5\,\text{kg}^2/(\text{cm}^4\,\text{Hz}),\quad q = 4{,}5\cdot 10^6\,\text{Hz},\quad K = 1\,\text{kg}^2/(\text{cm}^4\,\text{Hz}),\quad R = 1{,}0. \tag{6.27}$$

The realization of this process ($\ell_0$ = 10,000, $m_0$ = 3000) is presented in Fig. 5.

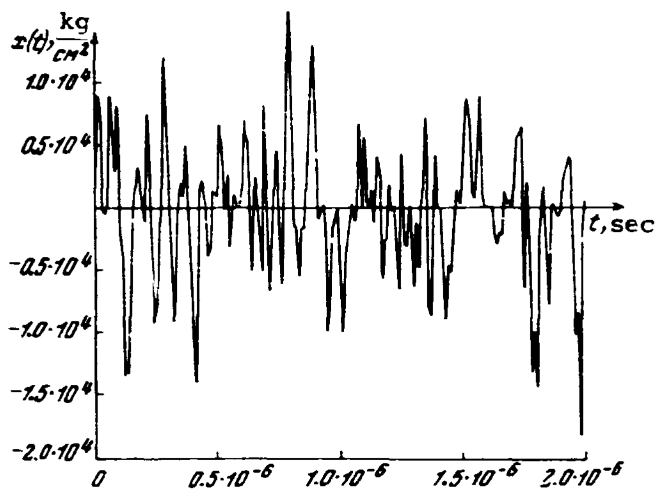


Fig. 5. Realization of model stationary "white" noise, with random amplitude and phase ($\Delta t=10^{-8}$ sec, $R=1$, $\mathfrak{D}=55\cdot 10^6$ Hz, $q=4.5\cdot 10^6$ Hz, $f_H=7.10^6$ Hz, $k=1$ kg$^2$/(cm$^4\cdot$Hz), $\ell_0=10{,}000$, $m_0=3000$).

Since only the values of $\mathfrak{D}$ and $f_H$ affect the form of the spectral density (4.22), from a comparison of (6.23) and (6.25-6.27), similarity of both processes in the spectral sense follows.

## 7. Basic Calculation Formulas

We consider the basic stages of obtaining estimates of intensity by the first formula (3.75), with the use of (3.16), (3.38) and (3.39), in the example of the realization: as before, let T be the /50
length of the realization, n the number of points and $\Delta t$ the digitization step. As has already been said, we assume that the mathematical expectation of the process equals zero. Therefore, the realization is centered beforehand. The Fourier transform of process X(t) is a complex function of the real variable f. The real part of the realization of complex random process S(f) is presented in Fig. 6 and, in Fig. 7, its imaginary part. The calculation formulas have the form:

$$\begin{gathered} \mathrm{Re}\ S(f) = \Delta t \sum_{j=1}^{n} x_j \cos[2\pi\Delta t(j - \tfrac{n+1}{2}) f] \ , \\ \mathrm{Im}\ S(f) = \Delta t \sum_{j=1}^{n} x_j \sin[2\pi\Delta t(j - \tfrac{n+1}{2}) f] \ , \\ -\frac{1}{2\Delta t} \le f \le \frac{1}{2\Delta t} \ . \end{gathered} \tag{7.1}$$

The real part Re S(f) is an even function of frequency, and the /51
imaginary part Im S(f) is odd. Therefore, the plot is made in Figs. 6 and 7, only for $f \ge 0$. It is easy to see that, approximately in the frequency range from f = 0 Hz to $f = 0.2\cdot 10^8$ Hz, and this means, in the frequency region $-F/2 \le f \le F/2$ ($F/2 \approx \Delta f_{eff}$), the process is close to stationary (uniform according to argument f) and, in the frequency region $f = \pm 0.2\cdot 10^8$ Hz, a nonstationary transition begins. We calculate the estimate of the spectral density, truncated in the interval [-F/2, F/2], of complex random process $S_F(f)$. The spectral density of the complex process, in distinction from the spectral density of the real process, is not

an even function. Therefore, it must be calculated over th entire interval from $t = -T/2$ to $t = T/2$. For our complex process $S(f)$, argument $t$ plays the role of frequency, in which the circumstance that $X(t)$ is assigned to the interval $[-T/2, T/2]$, and is equal to zero outside it, means that the spectral density of the process $S(f)$ equals zero outside this interval. In this manner, in accordance with the method of work [17], we select the digitization step $\delta f$ of process $S(f)$ from the condition

$$\delta f = \frac{1}{T} . \tag{7.2}$$

The digitization step $\delta f$ can be made smaller than $1/T$, but, for $\delta f = 1/T$, the volume of subsequent calculations is at a minimum. In our case, $T = 4 \cdot 10^{-6}$ sec, i.e., $\delta f = 0.25 \cdot 10^{6}$ Hz.

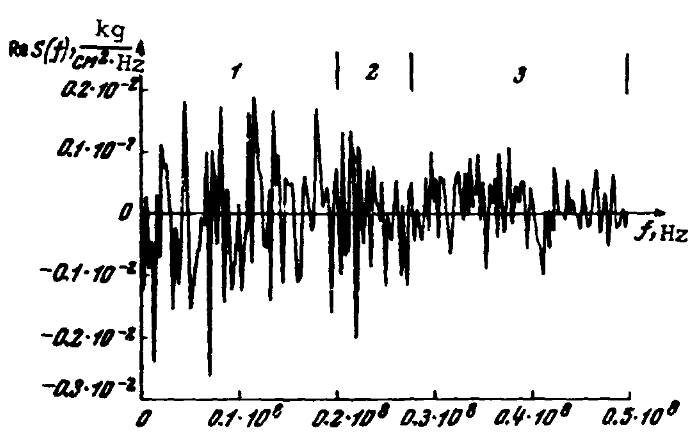


Fig. 6. Real part of realization of complex random process in frequency region: Re S(f) is an even function ($T = 4.0 \cdot 10^{-6}$ sec, $\delta f = 0.25 \cdot 10^{6}$ Hz); 1. stationary section; 2. transition section; 3. tail.

Let the number of points of the discrete complex process equal

$$p = F/\delta f , \tag{7.3}$$

then, the discrete Fourier transform of the truncated realization $S_{Fj}$ ($j = 1, 2, \ldots, p$) is written in the form

$$Re\, \tilde{S}_F(t) = \delta f \sum_{j=1}^{p} \left\{ Re\, S_{Fj} \cos\left[2\pi \delta f \left(j - \frac{p+1}{2}\right) t\right] + Im\, S_{Fj} \sin\left[2\pi \delta f \left(j - \frac{p+1}{2}\right) t\right] \right\} , \tag{7.4}$$

$$Im\, \tilde{S}_F(t) = 0 . \tag{7.5}$$

Since the imaginary part equals zero, $\tilde{S}_F(t) = Re\, \tilde{S}_F(t)$. The simplest estimate of the spectral density of process $S(f)$ and, consequently, the estimate of the intensity of process $X(t)$ will be

$$\varphi_{SF}(t) = \frac{1}{\Delta T} \int_{t-\Delta T/2}^{t+\Delta T/2} \frac{|\tilde{S}_F(\tau)|^2}{F} \, d\tau . \tag{7.6}$$

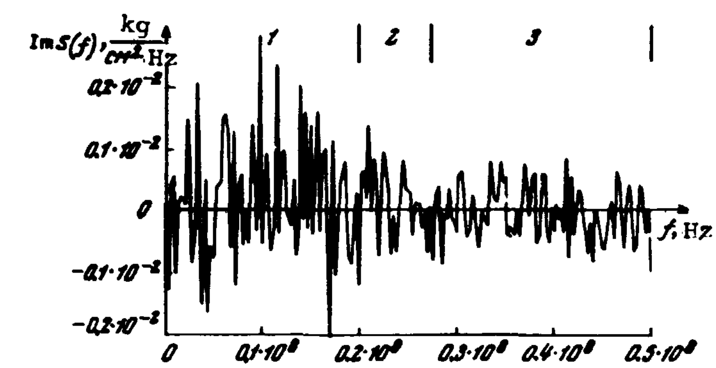


Fig. 7. Imaginary part of realization of complex random process in frequency region: $\mathrm{Im}\, S(f)$ is an odd function ($T = 4.0\cdot 10^{-6}$ sec, $\delta f = 0.25\cdot 10^{6}$ Hz); 1. stationary section; 2. transition section; 3. tail.

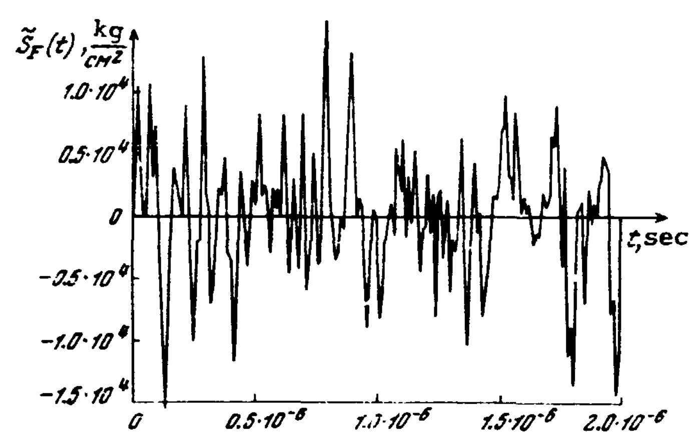


Fig. 8. Result of Fourier transform of realization of complex process from frequency region to time, at $F = 10^8$ Hz ($\Delta t = 10^{-8}$ sec).

Curves of functions $\tilde{S}_F(t)$, calculated for different F, are given in Figs. 8, 9 and 10. As follows from the results of section 3 (see 3.39 and 3.55), truncation of process S(f) in interval F and return to the time region are equivalent to filtration of initial process X(t), in the frequency band from $f = 0$ to $f = F/2$. This fact, in particular, is represented graphically in Figs. 8-10. In fact, with the maximum possible $F = 10^8$ Hz, $\tilde{S}_F(t)$ is simply the initial realization x(t) (see Fig. 5). With decrease in F, realization x(t) is more and more strongly smoothed.

The procedure described, for obtaining intensity estimates by formulas (7.1), (7.4) and (7.6) is very laborious, from the point of view of calculations. The calculations can be made economical, if the second formula is used for estimate (3.57). In this case, an analog of function $\tilde{S}_F(t)$ will be function

$$\hat{S}_F(t) = \frac{1}{\delta T}\int_{t-\delta T/2}^{t+\delta T/2} x(\xi)\, d\xi \ , \qquad (7.7)$$

obtained by sliding averages over interval /52
$\delta T$ of realization x(t), by means of the simplest smoothing operator, the effective filtration band of which is $1/2\ \delta T$, i.e.,

$$F \approx \frac{1}{\delta T} \qquad (7.8)$$

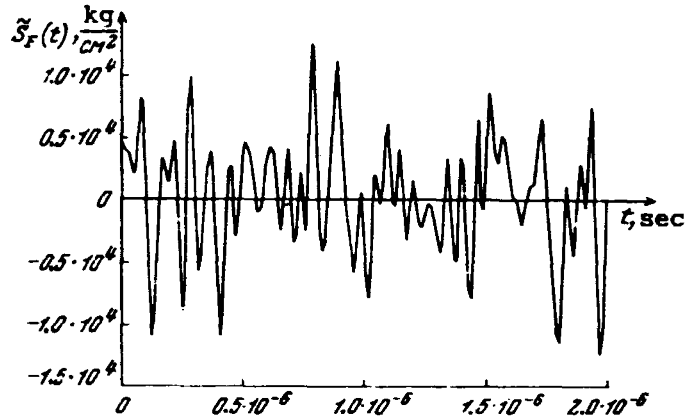


Fig. 9. Result of Fourier transform of realization of complex process from frequency region to time, for $F = 0.4 \cdot 10^8$ Hz ($\Delta t = 10^{-8}$ Hz).

$$\delta T \approx \frac{1}{2\Delta f_{eff}}. \qquad (7.9)$$

Finally, the second formula for the estimate (3.57) takes the form

$$\hat{\varphi}_{SF}(t) = \frac{\delta T}{\Delta T}\int_{t-\Delta T/2}^{t+\Delta T/2} \hat{S}_F^2(\xi)\,d\xi \,. \qquad (7.10)$$

For sequence $x_j$, we obtain

$$\hat{S}_j = \frac{1}{d}\sum_{l=1}^{d} x_{j+l-1} \,. \qquad (7.11)$$

$$j = 1, 2, 3, \dots, n-d+1,$$

$$\hat{\varphi}_k = \frac{\Delta t\, d}{g}\sum_{l=1}^{g} \hat{S}^2_{k+l-1} \,, \qquad (7.12)$$

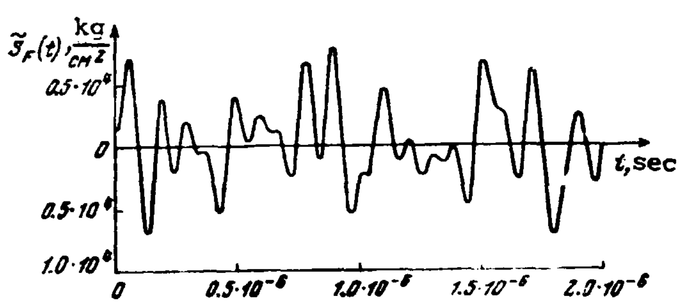


Fig. 10. Result of Fourier transform of realization of complex process from frequency region to time, for $F = 0.2 \cdot 10^8$ Hz ($\Delta t = 10^{-8}$ sec).

where /53

$$d = \delta T/\Delta t \,, \quad g = \Delta T/\Delta t \,,$$

$$k = 1, 2, 3, \dots, n-d-g+2.$$

Rapid calculation of sequences $\hat{S}_j$ and $\hat{\varphi}_k$ can be carried out in the following manner. For example, to calculate $\hat{S}_j$, we initially determine the mean

$$\hat{S}_1 = \frac{1}{d}\sum_{l=1}^{d} x_l \,, \qquad (7.13)$$

and, then, we use the recurrent relation

$$\hat{S}_j = \hat{S}_{j-1} - (x_{j-1} + x_{j+d-1})/d \qquad (7.14)$$

$$(j = 2, 3, 4, \dots, n-d+1).$$

A similar procedure can be used for calculation of $\hat{\phi}_k$.

We note that, if the bulk of the points of the initial realization $x_j$ were assigned in the time interval $[-T/2, T/2]$, the bulk of points of the intensity estimate $\hat{\phi}_k$ will be fixed in a smaller interval $[-T/2 + \delta t(d + g - 2)/2, T/2 - \Delta t(d + g - 2)/2]$.

Another way of increasing the speed of calculation is to use the FFT (fast Fourier transform) algorithm [21], for calculation of functions Re S(f), Im S(f) and $\tilde{S}_F(t)$, a modification of which, in the case of an arbitrary number n of values of a digital series, in distinction from the classical $n = 2^m$, is presented in the following section.

## 8. Modification of FFT (Fast Fourier Transform) Method

The procedure of calculation of estimates of the spectral density from one realization $S(f) = \mathrm{Re}\, S(f) + i\, \mathrm{Im}\, S(f)$ of complex, centered, stationary, random process S(f), is based on the Fourier transform $\tilde{S}_F(t)$, truncated to the interval [-F/2, F/2] of realization $S_F(t)$ /54

$$\tilde{S}_F(t) = \int_{-\infty}^{+\infty} S_F(f)\, e^{-i2\pi t f} df = e^{i2\pi t \frac{F}{2}} \int_0^F S(f - \tfrac{F}{2})\, e^{-i2\pi t f} df . \tag{8.1}$$

Actually, the simplest estimate of the spectral density has the form (see 7.6)

$$\varphi_{SF}(t) = \frac{1}{\Delta T} \int_{t-\Delta T/2}^{t+\Delta T/2} \frac{|\tilde{S}_F(\xi)|^2}{F}\, d\xi , \tag{8.2}$$

where the subintegral expression is the periodogram

$$I_{SF}(t) = \frac{|\tilde{S}_F(t)|^2}{F}$$

and $\Delta T$ is the averaging interval of the periodogram.

For calculations by computer, we have to deal with a continuous realization $S_F(f)$, and with the digital series

$$S_j = \mathrm{Re}\, S_F(\delta f j - \tfrac{F}{2}) + i\, \mathrm{Im}\, S_F(\delta f j - \tfrac{F}{2})$$

$j = 0, 1, 2, \ldots, p - 1$. The integral on the right side of (8.1) can be approximately replaced by the sum

$$\tilde{S}_{Fp}(t) = e^{i\pi t F} \delta f \sum_{j=0}^{p-1} S_j \exp(-i 2\pi t \delta f j) , \tag{8.3}$$

where $\delta f$ is the digitization step and $p = F/\delta f$. With correct selection of $\delta f$, $\tilde{S}_{Fp}(t)$ will approximate $\tilde{S}_F(t)$, with sufficient accuracy. (On selection of $\delta f$, see [17].) Thus, the estimate of the spectral density for the digital series takes the form

$$\varphi_{sp}(t) = \frac{1}{\Delta T} \int_{t-\Delta T/2}^{t+\Delta T/2} \frac{|\tilde{S}_{Fp}(\xi)|^2}{p\delta f} d\xi \ . \tag{8.4}$$

It is known that, at points $t = k/(p\delta f)$ ($k = 0, 1, 2, \ldots, [p/2]$), the quantities $\tilde{S}_{Fp}(k/(p\delta f)) = \tilde{S}_{pk}$ are the Fourier coefficients of digital series

$$\tilde{S}_{pk} = e^{i\pi k} \delta f \sum_{j=0}^{p-1} S_j \exp(-i 2\pi \tfrac{k}{p} j) \ .$$

An algorithm was proposed in work [21], based on recurrent calculation of the Fourier coefficient $\tilde{S}_{pk}$ of complex digital series $S_j$, through the Fourier coefficients of two auxiliary complex series, formed from the even and odd elements of series $S_j$, respectively (the Cooley-Tukey method). The number of operations necessary to obtain all the Fourier coefficients by this method is $\sim p \log_2 p$, in place of $p^2$ operations, by direct calculation with formula (8.5). For example, for $p = 1000$, the calculation time will be 100 times less, than by the usual calculations. This method also has the additional advantage that all the intermediate calculations and the final result are stored in the same internal memory cells as the initial values of the series. The Cooley-Tukey method is realized most simply, in the case, when the number of terms of series $S_j$ is an exponent of the number 2, $p = 2^m$, since the procedure of formation of the auxiliary series /55 can continue, until splitting of the even and uneven elements leads to complex series, consisting of one term. The Fourier transform of this term coincides with itself, multiplied by $\delta f$. Statement of the method and the standard programs written in Algol, for this case, are in work [12].

If p is not equal to an exponent of 2, a fast Fourier transform cannot be realized within the framework of a single algorithm. For example, in work [9], in this case, it is recommended that splitting into two series be carried out until it becomes easy to calculate the Fourier transform of the auxiliary series by the usual formula (8.5), or, if p has a simple divisor q, that the separation be carried out into q auxiliary series.

We consider a simple modification of the fast Fourier transform, which permits an estimate of the spectral density of type (8.4) to be obtained, for a digital series of arbitrary length p. Let $N = 2^m \geq p$, be the closest to the p power of 2. We add to initial series $S_j$ $(j = 0, 1, 2, \ldots, p-1)$ $N - p$ complex zero values, and we write the expression for the Fourier coefficients of the new series $\bar{S}_j$

$$\tilde{\bar{S}}_{pk} = e^{i\frac{\pi k p}{N}} \delta f \sum_{j=0}^{N-1} \bar{S}_j \exp\left(-i 2\pi \frac{k}{N} j\right) = = e^{i\frac{\pi k p}{N}} \delta f \sum_{j=0}^{p-1} S_j \exp\left(-i 2\pi \frac{k}{N} j\right), \quad k = 0, 1, 2, \ldots, N/2. \tag{8.6}$$

These coefficients can be calculated by means of the Cooley-Tukey method. It is evident that the values of $\bar{S}_{pk}$ are not Fourier coefficients of the initial series $S_j$, but are certain values of function $\hat{S}_{F_p}(t)$, at points $t = k/(N\delta f)$. In connection with this, we examine the question of approximate calculation of the integral in formula (8.4), by means of several values of the subintegral function.

Since the nodes of the function being integrated are fixed, construction of the rule of quadrature must be interpolated. We rewrite (8.4) in the form

$$\varphi_{s_p}(t) = \frac{1}{\Delta T} \int_0^{\Delta T} \frac{|\tilde{S}_{Fp}(\eta + t - \Delta T/2)|^2}{p\,\delta f}\, d\eta \tag{8.7}$$

and we write the approximate value of the integral, in the form of a quadrature sum

$$\int_0^{\Delta T}\frac{1}{\Delta T p\delta f}|\tilde{S}_{Fp}(\eta+t-\Delta T/2)|^2 d\eta \approx \sum_{\kappa=1}^{d} A_\kappa \frac{|\tilde{S}_{Fp}(\eta_\kappa+t-\Delta T/2)|^2}{\Delta T p\delta f}, \tag{8.8}$$

$$t-\frac{\Delta T}{2} \le \eta_\kappa < t+\frac{\Delta T}{2}, \quad \kappa = 1, 2, \ldots, d.$$

It follows from expression (8.3) that $\tilde{S}_{Fp}(t)$ is a continuous and continuously differentiable function and, consequently, the periodogram $I_{Sp}(t) = |\tilde{S}_{Fp}(t)|^2/(p\delta f)$ is a smooth function. We note from other properties of the periodogram that it is an oscillating function, with a maximum or minimum exchange period of approximately 2/F. In the interval $\Delta T$, it accomplishes a large number of oscillations $\Delta T \gg 2/F$ [17]. Considering the oscillatory nature and smoothness of the subintegral function, it is natural to adopt a system of trigonometric functions as the basis of interpolation and to use, as the approximation, a polynomial of the type

$$T_\ell(\eta) = a_0 + \sum_{\kappa=1}^{\ell}(a_\kappa \cos\frac{2\pi\kappa}{\Delta T}\eta + b_\kappa \sin\frac{2\pi\kappa}{\Delta T}\eta). \tag{8.9}$$

We select parameters $A_k$ and $\eta_k$, so that the rule of quadrature (8.8) gives a precise result for polynomials $T_\ell(\eta)$, of the highest possible degree. As was shown in [11], rule (8.8) cannot be precise, for any $A_k$ and $\eta_k$, for all trigonometric polynomials of degree d. The highest degree of accuracy, equal to d - 1, is achieved by the quadrature formula, with equal coefficients $A_k = \Delta T/d$ and equidistant nodes. In fact, it is easy to ascertain directly that the quadrature rule /56

$$\int_0^{\Delta T} I_{sp}(\eta+t-\Delta T/2)d\eta \approx \frac{\Delta T}{d}\sum_{\kappa=1}^{d} I_{sp}(\alpha+(\kappa-1)\frac{\Delta T}{d}+t-\frac{\Delta T}{2}), \tag{8.10}$$

where $\alpha$ is any number ($0 \le \alpha < \Delta T/d$), is exact, for all trigonometric polynomials of degree d - 1. It is sufficient for this, to verify that (8.10) is exactly fulfilled, for function $I_{Sp}(\xi) = \exp(i\,2\,\pi h\xi/\Delta T)$ ($h = 0, 1, \ldots, d-1$). If h = 0, the right and left sides obviously equal $\Delta T$. If $h = 1, 2, \ldots, d-1$, the integral equals

$$\int_0^{\Delta T}\exp[i2\pi h(\eta+t-\Delta T/2)/\Delta T]\,d\eta = \frac{\exp[i2\pi h(t-\Delta T/2)/\Delta T]}{i2\pi h/\Delta T}[e^{i2\pi h}-1]=0,$$

and the sum has the value

$$\frac{\Delta T}{d}\sum_{K=1}^{d} e^{i\frac{2\pi h}{\Delta T}(\alpha+t-\frac{\Delta T}{2})} e^{i\frac{2\pi h}{\Delta T}(K-1)\frac{\Delta T}{d}} = \frac{\Delta T}{d} e^{i\frac{2\pi h}{\Delta T}(\alpha+t-\frac{\Delta T}{2})} \frac{e^{id\frac{2\pi h}{d}}-1}{e^{i2\pi h/d}-1} = 0,$$

which proves exact fulfillment of (8.10). We note that the quadrature rule (8.10) contains an arbitrary parameter $\alpha$, the presence of which means that the equidistant nodes can be located completely arbitrarily in interval $\Delta T$.

For determination of the number of terms d in quadrature sum (8.10) which must be taken, in order to obtain the value of the integral with the required accuracy, an estimate of the residue must be carried out

$$R_d = \left| \int_0^{\Delta T} I_{sp}(\eta + t - \frac{\Delta T}{2})\, d\eta - \frac{\Delta T}{d}\sum_{K=1}^{d} I_{sp}(\alpha + (K-1)\frac{\Delta T}{d} + t - \frac{\Delta T}{2}) \right|. \qquad (8.11)$$

Since the value of $\alpha$ is completely arbitrary, for simplification of the contribution, we take it as the middle of the interval $[0, \Delta T/d]$

$$\alpha = \Delta T/(2d)\ . \qquad (8.12)$$

We will estimate the accuracy from the value of the maximum relative error

$$\varepsilon = R_d / \mathfrak{I}\ , \qquad (8.13)$$

where

$$\mathfrak{I} = \int_0^{\Delta T} I_{sp}(\eta + t - \Delta T/2)\, d\eta\ .$$

We reduce the periodogram to the form /57

$$I_{sp}(\xi) = \frac{\delta f}{P}\sum_{K=0}^{P-1} s_K s_K^* + \frac{2\delta f}{P}\sum_{j=1}^{P-1} Q_j(\xi)\ , \qquad (8.14)$$

where

$$Q_j(\tau) = \cos(2\pi\delta f\tau_j)\sum_{k=0}^{p-j-1} Re(S_k S_{k+j}^*) - \sin(2\pi\delta f\tau_j)\sum_{k=0}^{p-j-1} \mathrm{Im}(S_k S_{k+j}^*) .$$

By calculation of the integral, we obtain

$$\mathfrak{J} = \frac{\delta f\Delta T}{p}\sum_{k=0}^{p-1} S_k S_k^* + \frac{2\delta f\Delta T}{p}\sum_{j=1}^{p-1}\frac{\sin(\pi\delta f\Delta T j)}{\pi\delta f\Delta T j}Q_j(t) . \tag{8.15}$$

By substitution of (8.14) and (8.15) in (8.11) and by calculation of the sum, with allowance for (8.12) and introduction of similar terms, we obtain

$$R_d = \left|\frac{2\delta f\Delta T}{p}\sum_{j=1}^{p-1}\left(\frac{\pi a j}{\sin(\pi a j)} - 1\right)\frac{\sin(\pi\delta f\Delta T j)}{\pi\delta f\Delta T j}Q_j(t)\right| , \tag{8.16}$$

where $a = \delta f\Delta T/d$.

It follows from (8.16) that the smallness of $R_d$ is sufficient, so that

$$a(p-1) < 1 \quad \text{i.e.,} \quad d \gg \delta f\Delta T(p-1), \tag{8.17}$$

is satisfied, but such an estimate is too coarse and it does not take account of a whole series of properties of the subintegral function. Since $\sin(\pi a j)$ is in the denominator, the region of values in which the optimum d should be sought, is within

$$\delta f\Delta T(p-1) < d < \infty .$$

We assume that the real and imaginary parts of realization $S_j$ are bounded by the absolute value of constant M; then, the following inequality is true

$$\mathfrak{J} < 2\delta f\Delta T M^2 + \frac{4M^2}{\pi}\sum_{j=1}^{p-1}\frac{p-j}{pj}\varphi_j(t) ,$$

$$R_d < \left|\frac{4M^2}{\pi}\sum_{j=1}^{p-1}\left(\frac{\pi a j}{\sin(\pi a j)} - 1\right)\frac{p-j}{pj}\varphi_j(t)\right| ,$$

where

$$\mathfrak{P}_j(t) = \sin(\pi\delta f\Delta T j)\left(\cos(2\pi\delta f t_j) + \sin(2\pi\delta f t_j)\right) .$$

Since, for all j,

$$|\mathfrak{P}_j(t)| \leqslant \sqrt{2} ,$$

then,

$$\mathcal{J} < \left| 2\delta f\Delta T M^2 + \frac{4\sqrt{2}M^2}{\pi}\sum_{j=1}^{p-1}\frac{1}{j} - \frac{4\sqrt{2}M^2}{\pi}\frac{(p-1)}{p} \right| , \tag{8.18}$$

$$R_d < \left| \frac{4\sqrt{2}M^2}{\pi}\sum_{j=1}^{p-1}\left(\frac{\pi a j}{\sin(\pi a j)} - 1\right)\left(\frac{1}{j} - \frac{1}{p}\right) \right| . \tag{8.19}$$

By use of the known relationship /58

$$\sum_{j=1}^{p-1}\frac{1}{j} = \mathbf{C} + \ln(p-1) + \frac{1}{2(p-1)} - O\left(\frac{1}{p^2}\right) ,$$

where $\mathbf{C} \approx 0.577$ is the Euler constant, we obtain

$$\mathcal{J} < \left| 2\delta f\Delta T M^2 + \frac{4\sqrt{2}M^2}{\pi}\left(\mathbf{C} + \ln(p-1) + \frac{1}{2(p-1)} - 1 + \frac{1}{p}\right) \right| . \tag{8.20}$$

With expansion of $\pi aj/\sin(\pi aj)$ by powers $(aj)$ and by grouping the terms, we will have

$$\begin{aligned} R_d &< \left| \frac{4\sqrt{2}M^2}{\pi}\left\{ \frac{\pi^2 a^2}{6}\left(\sum_{j=1}^{p-1} j - \frac{1}{p}\sum_{j=1}^{p-1} j^2\right) + \frac{7\pi^4 a^4}{360}\left(\sum_{j=1}^{p-1} j^3 - \frac{1}{p}\sum_{j=1}^{p-1} j^4\right) + \ldots \right\} \right| = \\ &= \left| \frac{4\sqrt{2}M^2}{\pi}\left\{ \frac{\pi^2 a^2}{36}(p^2-1) + \frac{7\pi^4 a^4}{43200}(6p^4 - 10p^2 + 4) + \ldots \right\} \right| . \end{aligned} \tag{8.21}$$

A shortcoming of estimate of (8.20), connected with use of the expression $|\sin(\pi\delta f\Delta T j)| \leq 1$, is the circumstance that, with increase of p, it becomes worse, since it increases in proportion to the logarithm. However, since estimates (8.20) and (8.21) are constructed with identical assumptions, the estimate of their ratio $\varepsilon = R_d/\mathcal{J}$ will not become poorer (in the sense specified) with increase in p.

By extracting the main terms of estimates (8.20) and (8.21), we obtain an approximate expression for the maximum relative error

$$\varepsilon = \frac{R_l}{J} \approx \frac{\frac{4\sqrt{2}M^2}{\pi}\frac{\pi^2 a^2}{36}p^2}{\frac{4\sqrt{2}M^2}{\pi}\ln(p-1)} = \frac{\pi^2(\delta f)^2 \Delta T^2 p^2}{36\, d^2 \ln(p-1)} \tag{8.22}$$

In particular, if

$$d = \delta f \Delta T p$$

i.e., discrete values of $\tilde{S}_{Fp}(k/(p\delta f))$ are the Fourier coefficients, then

$$\varepsilon_f \approx \frac{\pi^2}{36 \ln(p-1)} \quad .$$

In this manner, for p = 1000, the relative error will not exceed 4%, which is completely acceptable for our purposes.

Returning to the question of calculation of the estimates of spectral density by means of expression (8.6), we note that, since, in this case

$$d = \delta f \Delta T N \geqslant \delta f \Delta T p \quad ,$$

the accuracy of integration will be no less, than in calculations, with the use of the Fourier coefficients (8.5), which had to be proved.

In conclusion of the section, we note that we have given the main attention to application of the FFT to calculation of estimates of spectral density (intensity) $\phi_{Sp}(t)$, assuming a fixed digital series $S_j$. It is clear that series $S_j$ can also be obtained by means of the FFT algorithm, by adding $(N_1 - n)$ zero values to centered time series $x_j$, where $N_1 = 2^{m_1} \geq n$, is the closest to the n power of 2. In this case, the digitization step of process S(f) does not increase over that which is necessary (see 7.2 and the following text) /53

$$\delta f = \frac{1}{N_1 \Delta t} \leqslant \frac{1}{T} \quad .$$

Examples are presented in the next section, of analysis of nonstationary, wideband, random processes. Digital calculation, of the estimates of the intensity are carried out, both by formula (7.6) (with the use of the modification of the fast Fourier transform method), and by formula (7.10), by means of programs, compiled in Algol language for the BESM-6 digital computer.

## 9. Results of Analysis of Model and Experimental Processes

We considered several digital calculations for stationary and nonstationary model, random processes, and we discussed certain questions which arise in analysis of actual (experimental) processes. All the basic characteristics of model processes ($\Delta t$, $\mathfrak{D}$, $q, f_H$, $R(t)$, $k$), their realizations ($\ell_0$, $m_0$), as well as the averaging intervals $F$, $\delta T$, $\Delta T$, $\Delta f$ and the theoretical intensity curves $b(t)$ and spectral densities $W_x(f)$ are presented in graphs. We elucidate first and foremost, how selection of parameters $F$ and $\Delta T$ affect the quality of intensity estimates (7.6). We will analyze a model stationary noise, a fragment of a realization of which is represented in Fig. 5.

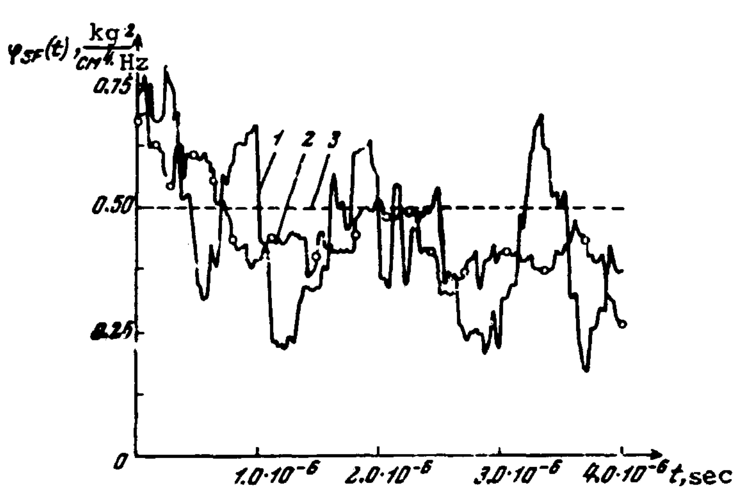


Fig. 11. Effect of change of averaging interval $\Delta T$ on estimate of intensity $\varphi_{SF}(t) = \tilde{S}_F^2(t)/F$ ($F = 0.4 \cdot 10^{+8}$ Hz); 1. $\Delta T = 0.41 \cdot 10^{-6}$ sec; 2. $\Delta T = 1.0 \cdot 10^{-6}$ sec; 3. theoretical intensity b.

The effect of change in parameters $F$ and $\Delta T$ on the nature of the estimates of intensity (7.6) are shown in Figs. 11 and 12. The estimate has a characteristic oscillatory appearance. With increase of $\Delta T$ and constant $F$, the oscillation amplitude decreases and the period increases. The oscillation period equals $\sim 2\ \Delta T$. With constant $\Delta T$, dispersion of the estimate increases with decrease in $F$. In order to preserve the optimum ratio between $F$ and $\Delta T$, as was shown in work [17] and in section 3, with increase of $F$, $\Delta T$ must be increased according to the pattern

$$\Delta T = \frac{\mathrm{const}}{F^{1/5}}, \quad (9.1)$$

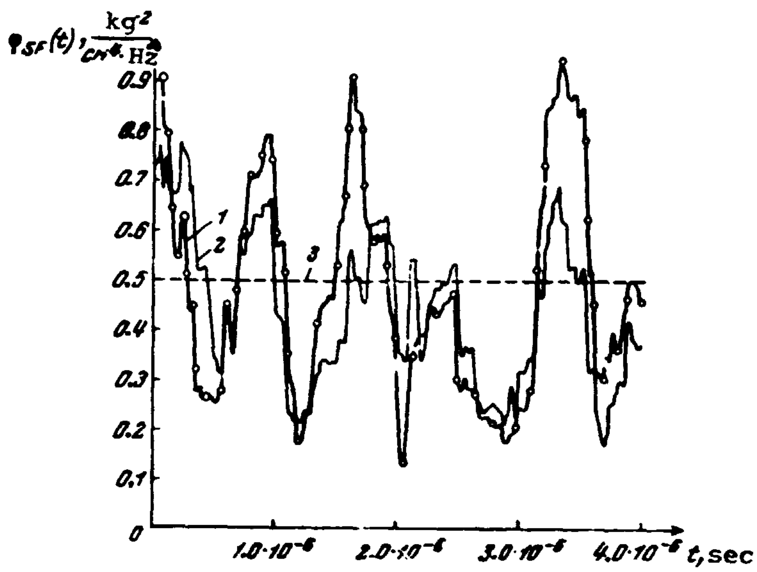


Fig. 12. Effect of change of interval F on estimate of intensity $\phi_{SF}(t) = \hat{S}^2_F(t)/F$ ($\Delta T = 0.41 \cdot 10^{-6}$ sec); 1. F = $0.2 \cdot 10^8$ Hz; 2. F = $0.4 \cdot 10^8$ Hz; 3. theoretical intensity b.

where the constant depends on the degree of unsteadiness of the intensity. In working with estimate (7.10), /60 to preserve the optimum ratio between $\delta T$ and $\Delta T$, with consideration of (7.8), we find that, with increase of $\delta T$, $\Delta T$ must be increased by the pattern

$$\Delta T = \mathrm{const}\,(\delta T)^{1/5} \quad (9.2)$$

A comparison of the two intensity estimates obtained by formulas (7.6) and (7.10) is demonstrated by the example of analysis of a nonstationary, wideband noise, the intensity of which is exponentially damped (Fig. 13). The small difference in the estimates is connected with the difference in /61 frequency characteristics of the filtration operators (3.38, 3.39 or 7.4) and (3.55 or 7.7), with wideband $\mathcal{L}$, $\mathcal{L} = F/2 \approx 1/(2\delta T)$. The frequency characteristic of filter (7.4) is variable over time t, in interval [-T/2, T/2]. It differs from rectilinear shape at the edges of interval [-T/2, T/2], and it is closest to rectilinear shape in the middle of the interval, in which, the larger T, the better the approximation. The analytical expression of the frequency characteristic of the filter has the form

$$\mathcal{H}_F(f,t) = \sqrt{P^2(f,t) + Q^2(f,t)}\,, \quad (9.3)$$

where

$$P(f,t) = \int_{t-T/2}^{t+T/2} \frac{\sin \pi F \tau}{\pi \tau} \cos 2\pi f \tau \, d\tau\,,$$

$$Q(f,t) = \int_{t-T/2}^{t+T/2} \frac{\sin \pi F \tau}{\pi \tau} \sin 2\pi f \tau \, d\tau\,.$$

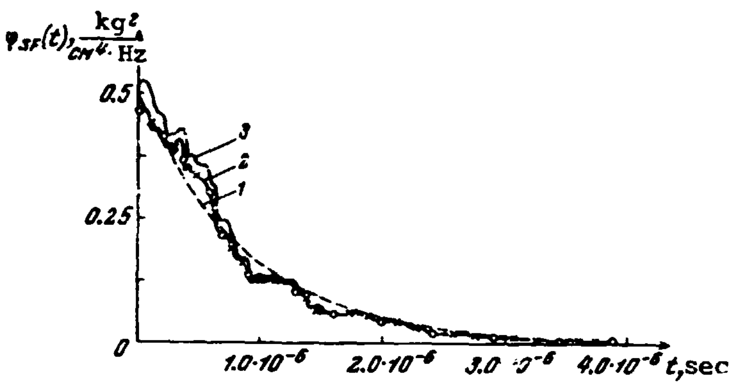


Fig. 13. Comparison of intensity estimates, obtained by means of various filtration operators with "wide" band $\mathcal{L} = F/2$ ($\Delta t = 10^{-8}$ sec; $\mathcal{D} = 55 \cdot 10^6$ Hz; $q = 4.5 \cdot 10^6$ Hz; $f_H = 7 \cdot 10^6$ Hz, $R(t) = \exp(-0.575 \cdot 10^6$ Hz$\cdot t$ sec); $k = 1$ kg$^2$/cm$^4$ $\cdot$Hz); $\Delta T = 1.0 \cdot 10^{-6}$ sec; $\ell_0 = 10{,}000$; $m_0 = 3000$); 1. theoretical intensity $b(t)$; 2. estimate $\hat{\phi}_{SF}(t)$ ($\delta T = 3 \cdot 10^{-8}$ sec); 3. estimate $\phi_{SF}(t)$ ($F = 0.4 \cdot 10^8$ sec).

The frequency characteristic of filter (7.7) is constant over time, and it does not depend on interval T

$$\mathcal{H}_F(f) = \frac{\sin(\pi \delta T f)}{\pi \delta T f} \quad (9.4)$$

An intensity estimate, calculated from realization of a nonstationary noise, the intensity of which increases exponentially, is presented in Fig. 14. In the interval $[-T/2 + (\Delta T + \delta T)/2,\ T/2 - (\Delta T + \delta T)/2]$, the estimate approximates the function sought $b(t)$ sufficiently well. Just as in the preceding example (Fig. 13), the intensity changes smoothly here in the observation interval and, consequently, for $\Delta T \sim T_x$, the estimate, not only has little dispersion, but little displacement. The characteristic scale of the problem

$$T_x \sim 2.0\ 10^{-6}\text{sec} - 4.0\ 10^{-6}\text{sec},$$

and the effective bandwidth of the noise

$$\Delta f_{eff} \sim 13.8 \cdot 10^6\ \text{Hz} - 19.0 \cdot 10^6\ \text{Hz},$$

consequently, the left side of condition (6.4) has this value

$$T_x \Delta f_{eff} \sim 28 - 76\ . \quad (9.5)$$

The practical conclusion can be drawn from this, that, if the /62
product of the characteristic scale of the unsteadiness and the effective bandwidth of the progress is within specified limits, estimate (7.5) or (7.10) will give satisfactory results.

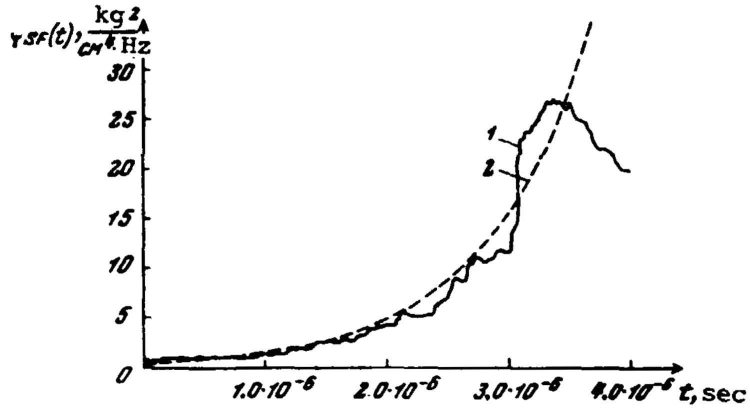


Fig. 14. Intensity estimate, calculated by one realization of model, nonstationary "white" noise ($\Delta t = 10^{-8}$ sec, $\mathfrak{D} = 55 \cdot 10^6$ Hz, $q = 4.5 \cdot 10^6$ Hz, $f_H = 7 \cdot 10^6$ Hz, $k = 1$ kg$^2$/(cm$^4$ Hz), $R(t) = \exp(0.575 \cdot 10^6$ Hz$\cdot$t sec), $\Delta T = 1.0 \cdot 10^{-6}$ sec, $\delta T = 3 \cdot 10^{-8}$ sec, $F = 0.333 \cdot 10^8$ Hz, $\ell_\Omega = 20{,}000$, $m_0 = 2000$); 1. estimate $\phi_{SF}(t)$; 2. theoretical intensity $b(t)$.

We now examine two examples of nonstationary, wideband noises, in which shifting of the intensity estimates plays the most significant role. Fragments of the realizations and the basic parameters of these processes are represented in Figs. 15 and 17. The intensity has a "bell-shaped" appearance (see Figs. 16 and 18). The width of the first "bell" $T_x \approx 2.0 \cdot 10^{-6}$ sec (we call this process slightly nonstationary), and the width of the second "bell" $T_x \approx 0.6 \cdot 10^{-6}$ sec (we call this process sharply nonstationary). As before, the effective bandwidth for both processes is

$$\Delta f_{eff} \sim 13.8 \; 10^6 \text{ Hz} - 19.0 \; 10^6 \text{ Hz}.$$

In this manner, for a slightly nonstationary process, /63

$$T_x \Delta f_{eff} \sim 28 - 38 , \quad (9.6)$$

and, for a sharply nonstationary one,

$$T_x \Delta f_{eff} \sim 8 - 11. \quad (9.7)$$

Intensity estimates, obtained for two independent realizations of a slightly nonstationary process, are

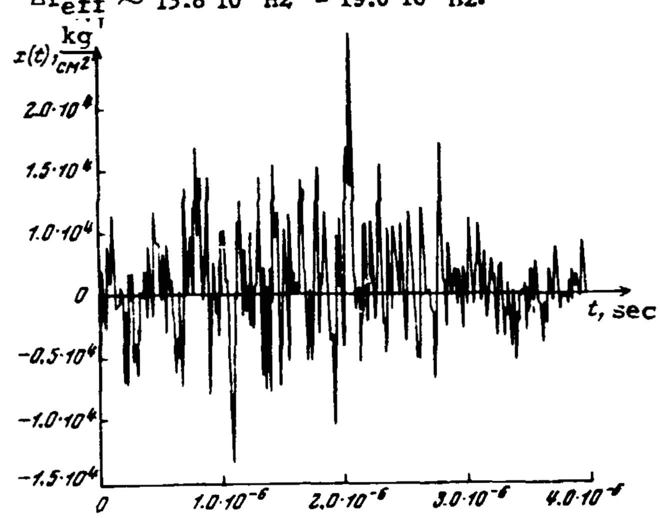


Fig. 15. Realization of model slightly nonstationary "white" noise ($\Delta t = 10^{-8}$ sec, $\mathfrak{D} = 55 \cdot 10^6$ Hz, $q = 4.5 \cdot 10^6$ Hz, $f_H = 7 \cdot 10^6$ H , $k = 1$ kg$^2$/(cm$^4 \cdot$Hz), $R(t) = 2.25 \cdot 10^{-12} / 2.25 \cdot 10^{-12} + (t - 1.5 \cdot 10^{-6})^2)$, $\ell_0 = 10{,}000$, $m_0 = 1$).

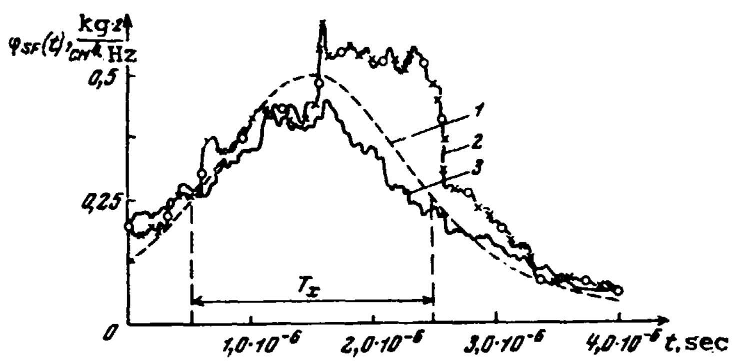


Fig. 16. Intensity estimates, calculated by two independent realizations of model slightly nonstationary "white" noise ($F = 0.4 \cdot 10^8$ Hz, $\Delta T = 1.0 \cdot 10^{-6}$ sec). The characteristic unsteadiness scale $T_x \approx 2 \cdot 10^{-6}$ sec; 1. theoretical intensity b(t); 2. $\ell_0 = 10{,}000$, $m_0 = 1$; 3. $\ell_1 = 15{,}000$, $m_1 = 1000$.

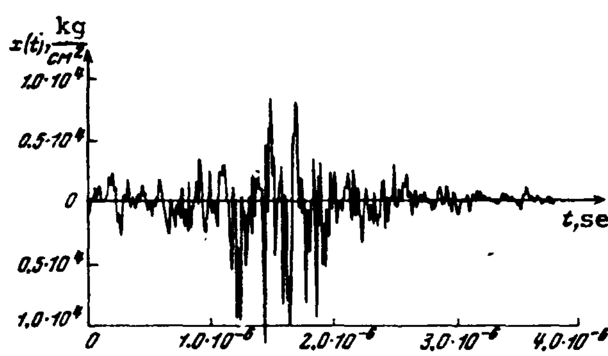


Fig. 17. Realization of model sharply nonstationary "white" noise ($\Delta t = 10^{-8}$ sec, $\mathfrak{D} = 55 \cdot 10^6$ Hz, $q = 4.5 \cdot 10^6$ Hz, $f_H = 7 \cdot 10^6$ Hz, $k = 1$ kg$^2$/(cm$^4$·Hz), $R(t) = 0.25 \cdot 10^{-12}/0.25 \cdot 10^{-12} + (t - 1.5 \cdot 10^{-6})^2)$, $\ell_0 = 20{,}000$, $m_0 = 2000$).

presented in Fig. 16. The estimates calculated on the realization ($\ell_1 = 15{,}000$, $m_1 = 1000$) approximates well the desired function, which is completely natural, since the process satisfies condition (9.5). However, the estimate calculated from realization ($\ell_0 = 10{,}000$, $m_0 = 1$) differs from the desired function on the right slope of the "bell." In examination of this realization (Fig. 15), it is easy to see that the error in the estimate is caused by the presence of a large scatter in the realization, in the middle of the observation interval. This error is due to dispersion of the estimate. The presence of large single discharges in some realizations of the random process means that process has slight ergodicity, relative to the correlation moment.

Two intensity estiments, calculated for the realizations represented in Fig. 17 ($\ell_0 = 21{,}000$, $m_0 = 2000$), are presented in Fig. 18. One estimate was obtained for the averaging interval $\Delta T = 1.0 \cdot 10^{-6}$ sec, of the large characteristic scale of unsteadiness $T_x = 0.6 \cdot 10^{-6}$ sec. The estimate has a large shift. The other estimate was calculated for the interval $\Delta T = 0.3 \cdot 10^{-6}$ sec, which is less than $T_x$. In this case, the displacement can be disregarded, but, in view of the fact that condition (9.5) is not observed, the estimate poorly approximates the desired function b(t), because of the great dispersion. /64

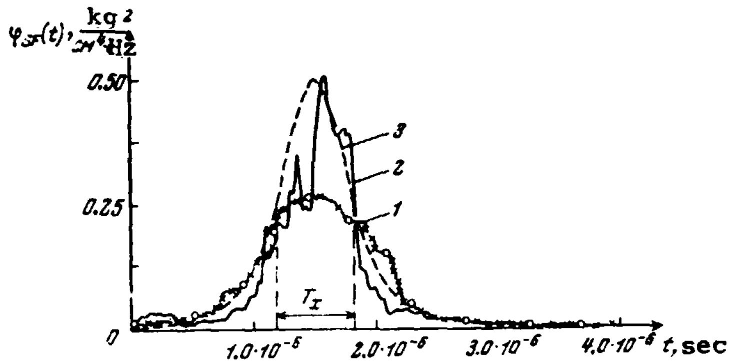

Fig. 18. Intensity estimates, obtained from realizations of model sharply nonstationary "white" noise ($F = 0.4 \cdot 10^8$ Hz). The characteristic scale of unsteadiness $T_x \approx 0.6 \cdot 10^{-6}$ sec; 1. $\Delta T = 1.0 \cdot 10^{-6}$ sec; 2. $\Delta T = 0.3 \cdot 10^{-6}$ sec; 3. theoretical intensity $b(t)$.

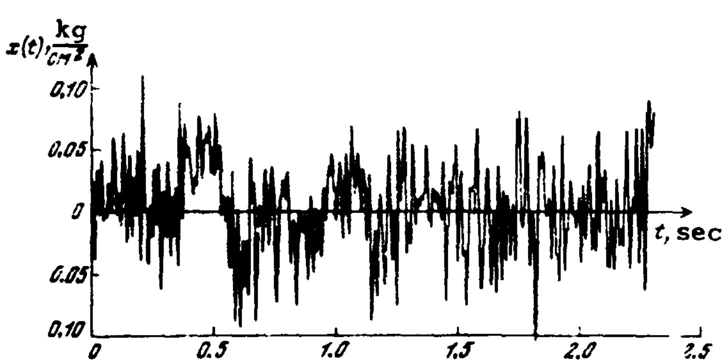

Fig. 19. Realization of wideband noise, obtained from experiment (experimental noise), $\Delta t = 0.005$ sec, $\Delta f_{eff} \approx 50$ Hz, $T_x \sim 0.5 - 1.0$ sec.

Before proceeding to analysis of a real process, a fragment of the realization of which, obtained from an experiment, is represented in Fig. 19, we note one important circumstance which provides new prospects for analysis of the intensity of real nonstationary processes. We have already noted in section 6 that the model processes constructed in this work can approximate actual random processes well. In order to be convinced of this, it is sufficient to glance at Fig. 15 and Fig. 19 and to agree that, if the legends of these figures were removed, it would be difficult to explain, why one realization was a realization of a model noise and the other was obtained in an experiment. The significant differences of the majority of real processes from the model ones considered here is that real processes can contain, first, narrowband sections which suddenly appear and then disappear, second, the spectrum of a real noise can have two or more different wideband levels in the interval from $f = 0$ Hz to $f = \Delta f_{eff}$. Finally, in real processes, a change in the effective bandwidth $\Delta f_{eff}$ can take place with the passage of time. Of course, interesting models can be constructed for all these cases, but we will not dwell on this. The concept of intensity loses meaning for such processes, since they begin to be wideband processes, which are close to white noise, but one can speak of the frequency average intensity by analogy with the average (over time) spectral density, which sometimes is used in analysis of nonstationary processes. Thus, in analysis of the realization of a nonstationary, random process, it is advisable, together with estimates of dispersion and the time average spectral density, to calculate the estimate of the frequency averaged intensity. With the exception of random coincidences, the nature of change in dispersion and intensity will be the same, if the unsteadiness of the noise is caused only by change in the total level, and is not connected with change in the effective bandwidth of the random process or with the appearance of narrow band components. A joint comparison of dispersion, frequency average intensity and time averaged spectral density of a nonstationary, random process, permits compilation of a representation of the structure of the process. As an example, we examine an analysis of a realization obtained experimentally. Estimates of the time average spectral density, calculated for two averaging intervals $\Delta f = 2.5$ Hz and $\Delta f = 20$ Hz, are presented in Fig. 20. The first estimate ($\Delta f = 2.5$ Hz) shows that there is an increased level in the process, in the 0 - 3 Hz/65 region, and a narrow band section in the 16 - 24 Hz region. It follows from the second estimate that two spectral density levels predominate in the frequency interval from $f = 0$ to $f = 50$ Hz: low-frequency ($0.075 \cdot 10^{-4}$ kg$^2$/(cm$^4 \cdot$Hz)) and high-frequency ($0.05 \cdot 10^{-4}$ kg$^2$/(cm$^4 \cdot$Hz)). Estimates of the dispersion and the frequency averaged intensity, calculated for this realization, are presented in Fig. 21. Although the estimates are plotted in different scales, they are such that the nature of change in them permits the location of the narrow band section to be identified immediately, namely: in the time interval from $t \approx 2.0$ sec to $t \approx 2.8$ sec, the equidistant change in dispersion and intensity is disrupted, i.e., in this section, the process begins to be wideband. In the time intervals from $t \approx 1.0$ sec to $t = 2.0$ sec and from $t \approx 2.8$ sec to $t \approx 3.6$ sec, analysis of the estimates of dispersion, intensity and spectral density ($\Delta f = 20$ Hz) shows that, in these sections, the low frequency noise level is decisive ($\sim 0.04 \cdot 10^{-4}$ kg$^2$/(cm$^4 \cdot$Hz)), and that the high frequency noise level ($\sim 0.025 \cdot 10^{-4}$ kg$^2$/(cm$^4 \cdot$Hz)) is decisive in the interval from $t \approx 3.6$ sec to $t \approx 4.9$ sec. We recall that the average spectral density level, by definition, is twice the average intensity level, i.e., the result of the spectral analysis (Fig. 20) and the result of analysis of intensity (Fig. 21) agree satisfactorily.

We note in conclusion that a more detailed analysis of the structure of this process can be conducted, by means of analysis of estimates of the instantaneous power spectrum.

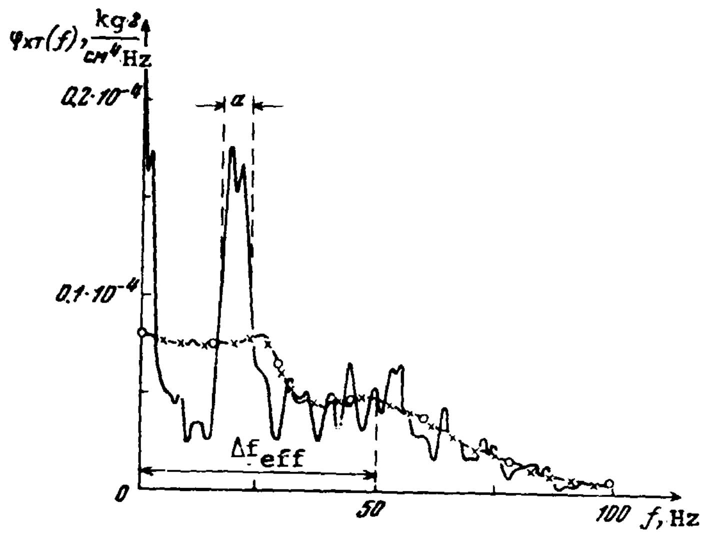


Fig. 20. Estimate of time averaged spectral density, obtained from realization of experimental noise with narrow band section (a) ($\Delta t$ = 0.0025 sec, T = 4.9 sec); 1. $\Delta f$ = 2.5 Hz; 2. $\Delta f$ = 20 Hz.

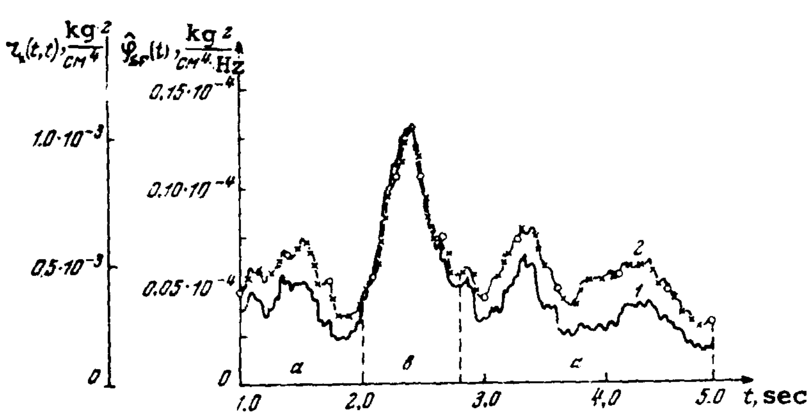


Fig. 21. Estimates of dispersion and frequency averaged intensity, calculated from realization of experimental noise with narrow band section ($\Delta t$ = 0.0025 sec, $\Delta f_{eff} \approx$ 50 Hz, $T_x \sim$ - 1.0 sec, $\delta T$ = 0.01 sec, F $\approx$ 50 Hz); a. wideband noise section; b. narrow band section; 1. intensity estimate $\hat{\phi}_{SF}(t)$ ($\Delta T \approx$ 0.3 sec); 2. dispersion estimate $r_x(t, t)$ ($\Delta T$ = 0.3 sec).

## Conclusions

1. Analysis of the connection between stationary (uniform) in the broad sense, random processes and white noise has permitted construction of a method of estimation of the intensity of nonstationary, wideband processes.

2. If random process X(t) is such that, in the time or frequency regions, it permits averaging over a large interval and, consequently, averaging over a small interval can be carried out by another variable, for such a process, the methods considered permit estimates of the quadratic characteristics to be obtained, with small displacement and dispersion.

3. By means of digital analysis of model random processes, especially realized by digital computer, and of real processes, the conditions of applicability of the methods have been studied in detail, and concrete recommendations for their practical use have been given.

4. The basic rule in calculation of intensity estimates from individual realizations is that the bandwidth F of the first filter must be

less than the effective bandwidth of the noise being studied, and the bandwidth $1/\Delta T$ of the second filter must be, first, much less than F and, second, the value of interval $\Delta T$ must be much less than the assumed characteristic scale of the nonstationary change of intensity $T_x$.

5. An estimate is proposed, for accelerated calculation of intensity, in which trigonometric functions are not used. The number of operations of the algorithm is proportional to the number of points of time series $x_j$.

6. A modification of the fast Fourier transform method is proposed, applicable to the calculation of estimates of the spectral density, for a digital series of arbitrary length n, in distinction from the classical case of $n = 2^m$.

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